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ON NICHOLS (BRAIDED) LIE ALGEBRAS

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We prove (i) Nichols algebra $\mathfrak{B}(V)$ of vector space V is finite-dimensional if and only if Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional; (ii) If the rank of connected V is 2 and $\mathfrak{B}(V)$ is an arithmetic root system, then $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$; and (iii) if $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there does not exist any m -infinity element with $p_{uu} \neq 1$ for any $u \in D(V)$, then $\dim(\mathfrak{B}(V)) = \infty$ if and only if there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$. Furthermore we give an estimation of dimensions of Nichols Lie algebras and two examples of Lie algebras which do not have maximal solvable ideals.

Keywords: Nichols Lie algebra, Nichols algebra, Nichols braided Lie algebra.

Mathematics Subject Classification 2000: 16W30, 16G10

1. Introduction

Nichols algebras play a prominent role in various areas of mathematics such as the theory of pointed Hopf algebras and logarithmic quantum field theories. Recently Heckenberger [6] established a one-to-one correspondence between arithmetic root systems and Nichols algebras of diagonal type as well as between twisted equivalence classes of arithmetic root systems and generalized Dynkin diagrams. A great deal of attention has been paid to the question of finite-dimensionality of Nichols algebras (see e.g. [2, 4, 5, 6, 7, 8, 15, 16]). The interest in this problem arose from the work of Andruskiewitsch and Schneider [1] on classification of finite dimensional (Gelfand-Kirillov) pointed Hopf algebras which are generalizations of quantized enveloping algebras of semi-simple Lie algebras. In [6] Heckenberger classified braided vector spaces of diagonal type with finite-dimensional Nichols algebras.

Let $\mathfrak{B}(V)$ be the Nichols algebra of vector space V . Let $\mathfrak{L}(V)$, $\mathfrak{L}^-(V)$ and $\mathfrak{L}_c(V)$ denote the braided Lie algebras generated by V in $\mathfrak{B}(V)$ under Lie operations $[x, y] = yx - p_{yx}xy$, $[x, y]^- = xy - yx$ and $[x, y]_c = xy - p_{xy}yx$, respectively, for any homogeneous elements $x, y \in \mathfrak{B}(V)$. $(\mathfrak{L}(V), [\])$, $(\mathfrak{L}^-(V), [\]^-)$ and $(\mathfrak{L}_c(V), [\]_c)$ are

called Nichols braided Lie algebra, Nichols Lie algebra and Nichols braided m -Lie algebra of V , respectively. It is clear that $(\mathfrak{L}(V), [\cdot])$ and $(\mathfrak{L}_c(V), [\cdot]_c)$ are equivalent as vector spaces. If $\mathfrak{B}(V)$ is finite dimensional then $\mathfrak{B}(V)$ is nilpotent, so $(\mathfrak{L}(V), [\cdot])$ and $(\mathfrak{L}^-(V), [\cdot]^-)$ also are nilpotent.

In [16], we studied the relationship between Nichols braided Lie algebras and Nichols algebras and provided a new method to determine when a Nichols algebra is finite dimensional. It was shown there that Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional if and only if Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional if there does not exist any m -infinity element in $\mathfrak{B}(V)$, and that $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$ does not hold if the rank of connected V is higher than 3 and $\mathfrak{B}(V)$ is an arithmetic root system.

In this paper we show that the condition “there does not exist any m -infinity element in $\mathfrak{B}(V)$ ” can be dropped and Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional if and only if Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional. Furthermore we prove that $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$ if the rank of connected V is equal to 2 and $\mathfrak{B}(V)$ is an arithmetic root system, and that $\dim(\mathfrak{B}(V)) = \infty$ if and only if there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$ if $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there does not exist any m -infinity element with $p_{uu} \neq 1$ for any $u \in D(V)$. Finally we give an estimation of dimensions of Nichols Lie algebras.

This paper is organized as follows. In section 2 we prove that Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional iff Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional. In section 3 we show that a generalized Dynkin diagram of V with rank n is a complete diagram with $q_{ii} \neq 1$ for any $1 \leq i \leq n$ iff $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$. This implies that if the rank of connected V is equal to 2 and $\mathfrak{B}(V)$ is an arithmetic root system, then $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$. In section 4 we show that if $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there does not exist any m -infinity element with $p_{uu} \neq 1$ for any $u \in D(V)$, then $\mathfrak{B}(V)$ is finite-dimensional iff $\mathfrak{L}^-(V)$ is finite-dimensional. We also give an estimation of dimensions of Nichols Lie algebras. In section 5 we find two examples of Lie algebras which have no maximal solvable ideals. In the appendix we prove that there does not exist any m -infinity element in Nichols algebra $\mathfrak{B}(V)$ with arithmetic root system $\Delta(\mathfrak{B}(V))$ over finite cyclic groups.

Throughout, $\mathbb{Z} =: \{x | x \text{ is an integer}\}$, $\mathbb{N}_0 =: \{x | x \in \mathbb{Z}, x \geq 0\}$, $\mathbb{N} =: \{x | x \in \mathbb{Z}, x > 0\}$. F denotes the base field of characteristic zero. $N_k := \text{ord}(p_{kk})$ when $p_{kk} \neq 1$; $N_k := \infty$ when $p_{kk} = 1$. $D = D(V) =: \{[u] \mid [u] \text{ is a hard super-letter}\}$. If $[u] \in D$ and $\text{ord}(p_{uu}) = m > 1$ with $h_u = \infty$, then $[u]$ is called an m -infinity element. Other notations are the same as in [2] and [16]. Throughout, braided vector space V is connected and of diagonal type with basis x_1, x_2, \dots, x_n and $C(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$, $n > 1$.

2. Relationship between Nichols algebras and Nichols Lie algebras

In this section we show that Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional if and only if Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional.

Lemma 2.1. *If homogeneous element $u \in \mathfrak{L}(V)$ and $\text{ord}(p_{uu}) > 2$, then $u^m \in \mathfrak{L}(V)$, $\forall m \in \mathbb{N}$.*

Proof. We prove this by induction on m . If $m = 1$, the claim holds obviously. Use the induction hypothesis, we have $u, u^2, \dots, u^m \in \mathfrak{L}(V)$. If $u^{m+1} \notin \mathfrak{L}(V)$, by Lemma 4.12 in [16] and $\begin{cases} [u, u^m] = (1 - p_{uu}^m)u^{m+1} \\ [u^2, u^{m-1}] = (1 - p_{uu}^{2(m-1)})u^{m+1} \end{cases}$, we know $1 - p_{uu}^m = 0, 1 - p_{uu}^{2(m-1)} = 0$, i.e. $p_{uu}^2 = 1$, which is a contradiction to $\text{ord}(p_{uu}) > 2$. \square

Theorem 2.2. *If $\mathfrak{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$, then $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)$ is finite-dimensional.*

Proof. If there does not exist any m -infinity element in $\mathfrak{B}(V)$, it follows from Theorem 4.11 in [16]. We now show that the theorem also holds if there exists an m -infinity element u in $\mathfrak{B}(V)$. In this case $\mathfrak{B}(V)$ is infinite-dimensional. If $\text{ord}(p_{uu}) = m > 2$, then $u^k \in \mathfrak{L}(V)$ by Lemma 3.1, for $\forall k \in \mathbb{N}$. Consequently, $\mathfrak{L}(V)$ is infinite-dimensional. Assume that $p_{uu}^2 = 1$ and $\mathfrak{L}(V)$ is finite-dimensional, then we know $\exists v \in D$ such that $p_{uv}p_{vu} \neq 1$ with $v \neq u$ by Proposition 4.10 in [16]. We prove $u^k v \in \mathfrak{L}(V)$ by induction on k . If $k = 1$, the claim holds by Lemma 4.12 in [16]. Using the induction hypothesis we have $u^k v \in \mathfrak{L}(V)$. If $u^{k+1}v \notin \mathfrak{L}(V)$, then $p_{uu}^{2k}p_{uv}p_{vu} = 1$ by Lemma 4.12 in [16]. which is a contradiction to $p_{uu}^2 = 1$ and $p_{uv}p_{vu} \neq 1$. This has shown $u^k v \in \mathfrak{L}(V)$ for any $k \in \mathbb{N}$. It is clear that $\{u^k v \mid k \in \mathbb{N}\}$ or $\{vu^k \mid k \in \mathbb{N}\}$ is a subset of restricted PBW basis. Consequently, $\mathfrak{L}(V)$ is infinite-dimensional, which is a contradiction. \square

3. Conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

In this section we prove that a generalized Dynkin diagram of V with rank n is a complete diagram with $q_{ii} \neq 1$ for any $1 \leq i \leq n$ if and only if $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$. This implies that if the rank of connected V is equal to 2 and $\mathfrak{B}(V)$ is an arithmetic root system, then $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

Lemma 3.1. *Set $\tilde{p}_{uv} := p_{uv}p_{vu}$. Let a, b, c, d, e, f denote $1 - \tilde{p}_{vw}, 1 - \tilde{p}_{uw}, 1 - \tilde{p}_{vu}, 1 - \tilde{p}_{vu}\tilde{p}_{uw}, 1 - \tilde{p}_{vu}\tilde{p}_{vw}, 1 - \tilde{p}_{vw}\tilde{p}_{uw}$, respectively. If homogeneous elements $u, v, w \in \mathfrak{L}(V)$, then $uvw, uuv, vuv, vuw, wuv, wvu \in \mathfrak{L}(V)$ if one of the following is fulfilled:*

- (i) $a \neq 0, b \neq 0$;
- (ii) $a \neq 0, c \neq 0$;
- (iii) $a \neq 0, d \neq 0$;
- (iv) $b \neq 0, c \neq 0$;
- (v) $b \neq 0, e \neq 0$;
- (vi) $c \neq 0, f \neq 0$.

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Proof. Analogously to a formula of Kharchenko on braided commutators in [11], the following equation holds: $[[u, v], w] = [u, [v, w]] + p_{uv}[[u, w], v] + p_{vu}(p_{uv}p_{vu} - 1)[u, w]v$.

(i) Assume now that $b \neq 0$ and $a \neq 0$. Then $[u, w]v \in \mathfrak{L}(V)$ and hence $v[u, w] \in \mathfrak{L}(V)$ by this formula since $a \neq 0$. If $c \neq 0$, then $[w, u]v \in \mathfrak{L}(V)$ and $v[w, u] \in \mathfrak{L}(V)$. there exist $\alpha, \beta, \alpha', \beta' \in F$ such that $wu = \alpha[u, w] + \beta[w, u]$ and $uw = \alpha'[u, w] + \beta'[w, u]$ since $b \neq 1$. Consequently, $uwv = \alpha[u, w]v + \beta[w, u]v \in \mathfrak{L}(V)$ and $wuv = \alpha'[u, w]v + \beta'[w, u]v \in \mathfrak{L}(V)$. Similarly, $uvw, vuw, vwu, wvu \in \mathfrak{L}(V)$. If $c = 0$, then $d \neq 0$ and $e \neq 0$, then $(uw)v, (wu)v, v(uw), v(wu), (wv)u, u(vw) \in \mathfrak{L}(V)$ by Lemma 4.12 in [16].

Similarly, we can prove (ii) and (iv).

(iii) Assume now that $a \neq 0$ and $d \neq 0$. Then $b \neq 0$ or $c \neq 0$. In the first case (iii) follows from (i). In the second, (iii) also follows from (ii).

Similarly, we can prove (v) and (vi). \square

Lemma 3.2. Assume that homogeneous elements u and v are in $\mathfrak{L}(V)$.

(i) Then $uv \in \mathfrak{L}(V) \iff vu \in \mathfrak{L}(V)$.

(ii) If $p_{uv}p_{vu} \neq 1$ and $p_{uu}^2 = 1$, then $u^m v, u^{m-1}vu, \dots, vu^m \in \mathfrak{L}(V), \forall m \in \mathbb{N}$.

(iii) Assume that $p_{uu}^2 \neq 1$ and that $u, v, uv \in \mathfrak{L}(V)$. Then $u^k v u^l \in \mathfrak{L}(V)$ for any $k, l \in \mathbb{N}_0$.

(iv) If $p_{uv}p_{vu} \neq 1$ or $p_{uu}^2 \neq 1$ with $uv \in \mathfrak{L}(V)$, then $u^m v, u^{m-1}vu, \dots, vu^m \in \mathfrak{L}(V), \forall m \in \mathbb{N}$.

Proof. (i) It is clear.

(ii) We prove this by induction on m . set $u := u, v := u$ and let $w := u^{m-2}v, u^{m-3}vu, \dots$,

$vu^{m-2}, m \geq 2$, respectively. then $a = 1 - p_{uu}^{2(m-2)}p_{uv}p_{vu} \neq 0, b = 1 - p_{uu}^{2(m-2)}p_{uv}p_{vu} \neq 0$, then $u^m v, u^{m-1}vu, \dots, vu^m \in \mathfrak{L}(V)$ by Lemma 3.1.

(iii) The proof can be done by induction on $k+l$. (i) and the induction hypothesis imply that it suffices to prove that $u^m v \in \mathfrak{L}(V)$ for any $m \in \mathbb{N}$. This again is done by induction on m . The claim holds by assumption for $m = 1$.

Let $m \geq 1$ such that $u^k v \in \mathfrak{L}(V)$ for any $k \in \{0, 1, \dots, m\}$. Since $p_{uu}^2 \neq 1$, Lemma 2.1 implies that $u^k \in \mathfrak{L}(V)$ for any $k \geq 1$. If $p_{u^m v, u}p_{u, u^m v} \neq 1$, then $u^{m+1}v \in \mathfrak{L}(V)$. By the same reason, if $p_{u^{m-1}v, u^2}p_{u^2, u^{m-1}v} \neq 1$, then $u^{m+1}v \in \mathfrak{L}(V)$. If both inequalities fail, then $p_{uv}p_{vu} = p_{uu}^{-2m}$ and $p_{uu}^4 = 1$. Since $p_{uu}^2 \neq 1$, we conclude that $p_{uu}^2 = -1$ and $p_{uv}p_{vu} = (-1)^{-m}$.

Assume now that m is odd. Apply Lemma 3.1 to the triple (u, u^m, v) . Since $p_{u, u^m}p_{u^m, u} = p_{uu}^{2m} = -1$ and $p_{uv}p_{vu} = -1$, Lemma 3.1 implies that $u^{m+1}v \in \mathfrak{L}(V)$.

Assume now that m is even. Then $p_{uv}p_{vu} = 1$ and $m \geq 2$. Apply Lemma 3.1 to the triple (u, u^{m-1}, uv) . Since $p_{u, u^{m-1}}p_{u^{m-1}, u} = p_{uu}^{2m-2} = -1$ and $p_{u, uv}p_{uv, u} = p_{uu}^2 p_{uv}p_{vu} = -1$, Lemma 3.1 implies that $u^{m+1}v \in \mathfrak{L}(V)$. This proves the claim.

(iv) It is clear by (ii) and (iii). \square

Lemma 3.3. Let $m \in \mathbb{N}$ and for any $1 \leq i \leq m$ let w_i be a homogeneous element

in $\mathfrak{L}(V)$ with $w_i^2 = 0$ when $p_{w_i, w_i}^2 = 1$. If $p_{w_i, w_j} p_{w_j, w_i} \neq 1$ when $w_i \neq w_j$ for any $1 \leq i \neq j \leq m$, then $W := \prod_{i=1}^m w_i \in \mathfrak{L}(V) \oplus F$.

Proof. It is clear for $m = 1$. For $m = 2$, if $w_1 = w_2$ and $p_{w_1, w_2} p_{w_2, w_1} = 1$, then $p_{w_1, w_1}^2 = 1$ and $w_1^2 = w_1 w_2 = 0 \in \mathfrak{L}(V) \oplus F$. If $p_{w_1, w_2} p_{w_2, w_1} \neq 1$, then $w_1 w_2 \in \mathfrak{L}(V) \oplus F$ by Lemma 4.12 in [16].

Assume $m > 2$. If the claim does not hold, then there exists a minimal t such that $W := \prod_{i=1}^t w_i \notin \mathfrak{L}(V)$. Obviously, $t > 2$, $W \neq 0$ and $w_1 w_2 \cdots w_{t-2}, w_1 w_2 \cdots w_{t-1} \in \mathfrak{L}(V)$. Let $u := w_{t-1}$, $v := w_t$, $w := w_1 w_2 \cdots w_{t-2}$. If $u = v$ and $p_{uv} p_{vu} = 1$, then $p_{uu}^2 = 1$, which implies $uu = uv = 0$. This contradicts to $W := \prod_{i=1}^t w_i \notin \mathfrak{L}(V)$. Consequently, $p_{uv} p_{vu} \neq 1$. It follows from Lemma 4.12 in [16] that $\tilde{p}_{v, wu} = 1$. By Lemma 3.1, we have that $\tilde{p}_{v, w} = 1$, which implies $\tilde{p}_{uv} = 1$. This is a contradiction. Here $\tilde{p}_{uw} := p_{uw} p_{wv}$. \square

Lemma 3.4. Assume that $m \in \mathbb{N}$ and w_i is a homogeneous element in $\mathfrak{L}(V)$ with $w_i^2 = 0$ when $p_{w_i, w_i}^2 = 1$ for any $1 \leq i \leq m$. If $p_{w_i, w_j} p_{w_j, w_i} \neq 1$ when $w_i \neq w_j$ for any $1 \leq i \neq j \leq m$, then $W := \prod_{i=1}^m w_i^{a_i} \in \mathfrak{L}(V) \oplus F$ for any $a_i \in \mathbb{N}_0$.

Proof. We show this by following several steps.

- (i) Set $N(W) := \sum_{i=1}^m a_i$. We show $W \in \mathfrak{L}(V) \oplus F$ by induction on $N(W)$. It is clear when $W = 0$. We assume $W \neq 0$ from now on. When $N(W) = 0$, $W = 1 \in \mathfrak{L}(V) \oplus F$. When $N(W) = 1$, $W \in \mathfrak{L}(V) \oplus F$. Now assume $N(W) > 1$.
- (ii) If $m = 1$, $W = w_1^{a_1} \in \mathfrak{L}(V) \oplus F$ by Lemma 2.1.
- (iii) If $a_i = 1$ for $1 \leq i \leq m$, then $W \in \mathfrak{L}(V) \oplus F$ by Lemma 3.3.
- (iv) If there exists $1 \leq i_0 \leq m$ such that $a_{i_0} > 1$, let $z_1 = w_{i_0}$, $z_2 = w_{i_0+1}^{a_{i_0}+1} \cdots w_m^{a_m} w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0}-1}$. By induction hypothesis and Lemma 3.2 (i), $z_1, z_2, z_1 z_2 \in \mathfrak{L}(V)$. By Lemma 3.2(iv), $W' = z_1^{a_{i_0}} z_2 \in \mathfrak{L}(V)$. Using induction again, we have $w_{i_0}^{a_{i_0}} \cdots w_m^{a_m} w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0}-1} \in \mathfrak{L}(V)$. By Lemma 3.2 (i) we have that $W = w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0}-1} w_{i_0}^{a_{i_0}} \cdots w_m^{a_m} \in \mathfrak{L}(V)$. \square

A generalized Dynkin diagram is called a complete diagram if there exists an edge between any two vertexes.

Theorem 3.5. A generalized Dynkin diagram of V with rank n is a complete diagram with $q_{ii} \neq 1$ for any $1 \leq i \leq n$ if and only if $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

Proof. The necessity. Let $w_i \in \{x_1, x_2, \dots, x_n\}$ for any $1 \leq i \leq m$. Then $p_{w_i, w_j} p_{w_j, w_i} \neq 1$ when $w_i \neq w_j$ for any $1 \leq i \neq j \leq m$ and $\prod_{i=1}^m w_i^{a_i} \in \mathfrak{L}(V) \oplus F$ for any $a_i \in \mathbb{N}_0$ by Lemma 3.4. Consequently, $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

The sufficiency. Obviously $q_{ii} \neq 1$ since $x_i^2 \in F \oplus \mathfrak{L}(V)$ for $1 \leq i \leq n$. By Lemma 5.2 in [16], we have that the generalized Dynkin diagram of V is a complete diagram. \square

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We immediately have

Corollary 3.6. *Assume that $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and V is connected. Then $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$ if the rank of V is 2.*

Lemma 3.7. *Assume that w_i is a homogeneous element in $\mathfrak{L}(V)$ with $w_i^2 = 0$ when $p_{w_i, w_i}^2 = 1$ for any $1 \leq i \leq m$. If $\prod_{j=1}^r w_{i_j} \in \mathfrak{L}(V) \oplus F$ for any distinct $1 \leq i_2, i_2, \dots, i_r \leq m$, then $W := \prod_{i=1}^m w_i^{a_i} \in \mathfrak{L}(V) \oplus F$ for any $a_i \in \mathbb{N}_0$.*

Proof. Set $N(W) := \sum_{i=1}^m a_i$. We show $W \in \mathfrak{L}(V) \oplus F$ by induction on $N(W)$. It is clear when $W = 0$. We assume $W \neq 0$ from now on. When $N(W) = 0$, $W = 1 \in \mathfrak{L}(V) \oplus F$. When $N(W) = 1$, $W \in \mathfrak{L}(V) \oplus F$. Now assume $N(W) > 1$.

(ii) If $m = 1$, $W = w_1^{a_1} \in \mathfrak{L}(V) \oplus F$ by Lemma 2.1.

(iii) If $a_i = 1$ for $1 \leq i \leq m$, then $W \in \mathfrak{L}(V) \oplus F$ by assumption.

(iv) If there exists $1 \leq i_0 \leq m$ such that $a_{i_0} > 1$, let $z_1 = w_{i_0}, z_2 = w_{i_0+1}^{a_{i_0+1}} \cdots w_m^{a_m} w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0-1}}$. By induction hypothesis and Lemma 3.2 (i), $z_1, z_2, z_1 z_2 \in \mathfrak{L}(V)$. By Lemma 3.2(iv), $z_1^{a_{i_0}} z_2 \in \mathfrak{L}(V)$. Using induction again, we have $w_{i_0}^{a_{i_0}} \cdots w_m^{a_m} w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0-1}} \in \mathfrak{L}(V)$. By Lemma 3.2 (i) we have that $W = w_1^{a_1} \cdots w_{i_0-1}^{a_{i_0-1}} w_{i_0}^{a_{i_0}} \cdots w_m^{a_m} \in \mathfrak{L}(V)$. \square

Lemma 3.8. *If homogeneous element u in $\mathfrak{L}(V)$ and $p_{u,u} \neq 1$, then $u^2 \in \mathfrak{L}(V)$.*

Proof. $[u, u] = u^2 - p_{u,u} u^2 = (1 - p_{u,u}) u^2$, which implies $u^2 \in \mathfrak{L}(V)$. \square

Proposition 3.9. *Assume that u and v are homogeneous elements in $\mathfrak{L}(V)$ and $(p_{uv, uv})^2 \neq 1$ with $uv, uvu, vuv \in \mathfrak{L}(V)$. If $u^2 = 0$ when $p_{uu}^2 = 1$ and $v^2 = 0$ when $p_{vv}^2 = 1$, then $W = \prod_{i=1}^m u^{a_i} v^{b_i} \in \mathfrak{L}(V)$ for any $a_i, b_i \in \mathbb{N}_0$, $m \in \mathbb{N}$.*

Proof. Set $N(W) := \sum_{i=1}^m (a_i + b_i)$. We show $W \in \mathfrak{L}(V) \oplus F$ by induction on $N(W)$. When $N(W) = 0$, $W = 1 \in \mathfrak{L}(V) \oplus F$. When $N(W) = 1$, $W \in \mathfrak{L}(V) \oplus F$. Assume $N(W) > 1$. Set $c_{2i-1} = a_i, c_{2i} = b_i, y_{2i-1} = u, y_{2i} = v$ for $1 \leq i \leq m$.

(i) $(uv)^m u, (vu)^m v, (vu)^m, (uv)^m \in \mathfrak{L}(V)$ for any $m \in \mathbb{N}$. In fact, $(vu)^m, (uv)^m \in \mathfrak{L}(V)$ for any $m \in \mathbb{N}$ by Lemma 2.1. $(uv)^m u, (vu)^m v \in \mathfrak{L}(V)$ by Lemma 3.2 (iv).

(ii) By Lemma 2.1, $y_i^{c_i} \in \mathfrak{L}(V)$ for $1 \leq i \leq m$. Now we assume that there are two non-zero elements in $\{c_i \mid 1 \leq i \leq m\}$.

(iii) If there exists $1 \leq i_0 \leq 2m$ such that $c_{i_0} > 1$, let $z_1 = y_{i_0}, z_2 = y_{i_0+1}^{c_{i_0+1}} \cdots y_{2m}^{c_{2m}} y_1^{c_1} \cdots y_{i_0-1}^{c_{i_0-1}}$. By induction, $z_1, z_2, z_1 z_2 \in \mathfrak{L}(V)$. By Lemma 3.2(iv), $W' = z_1^{c_{i_0}} z_2 \in \mathfrak{L}(V)$. Using induction again, we have $W \in \mathfrak{L}(V)$. \square

Proposition 3.10. *V is a quantum linear space if and only if $\text{span}\{x_i^{a_i} \mid 0 < a_i < N_i, 1 \leq i \leq n\} = \mathfrak{L}(V)$.*

Proof. The sufficiency. It is clear $[x_i, x_j] = 0$ for any $i \neq j$, i.e. $p_{ij} p_{ji} = 1$ for any $i \neq j$. Consequently, V is a quantum linear space. The necessity. It is clear $x_i^{a_i} \in \mathfrak{L}(V)$. It is enough to show that $b(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = 0$ when there exist two

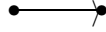
elements in x_{i_1}, \dots, x_{i_r} are different, where $b(x_{i_1}, x_{i_2}, \dots, x_{i_r})$ denote a method of adding bracket on $x_{i_1}, x_{i_2}, \dots, x_{i_r}$. We show this by induction on r . It is clear when $r = 2$. When $r > 2$, let $b(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = [u_1, u_2]$ with $u_1 = b(x_{i_1}, \dots, x_{i_s})$. If $u_1 = 0$ or $u_2 = 0$, then $b(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = 0$. If $u_1 \neq 0$ and $u_2 \neq 0$, then $x_{i_1} = \dots = x_{i_s}$ and $x_{i_{s+1}} = \dots = x_{i_r}$ by induction. It follows $b(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = 0$ from braided Jacobi identity. \square

4. Nichols Lie algebra $\mathfrak{L}^-(V)$

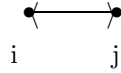
In this section it is proved that if $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there does not exist any m -infinity element with $p_{u,u} \neq 1$ for any $u \in D(V)$ then $\dim(\mathfrak{B}(V)) = \infty$ if and only if there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$. Furthermore, we give an estimation of dimensions of Nichols Lie algebras.

If A is an associative algebra and we define $[a, b]^- = ab - ba$ for any $a, b \in A$, then $(A, [\]^-)$ is a Lie algebra, which is written as A^- . Recall that Nichols Lie algebra $\mathfrak{L}^-(V)$ is the Lie algebra generated by V in $\mathfrak{B}(V)^-$ (see [16]).

For braided vector space V , we can define a directed diagram as follows. We use an arrow from i to j to denote $p_{ij} \neq 1$:



We use a bi-arrow between i and j to denote $p_{ij} \neq 1$ and $p_{ji} \neq 1$:



This directed diagram is called a directed generalized Dynkin diagram. Furthermore, we add a line between i and j when $p_{ij}p_{ji} \neq 1$. In this case, we call this diagram a mixed generalized Dynkin diagram. We can study Nichols Lie algebras by means of the directed generalized Dynkin diagrams.

Let $l_u^0[v] := [v]$, $l_u^i[v] := [[u], l_u^{i-1}[v]]$, $r_u^0[v] := [v]$, $r_u^i[v] := [r_u^{i-1}[v], [u]]$, $i \geq 1$ for any $u, v \in \mathfrak{B}(V)$. similarly, Let $\bar{l}_u^0[v]^- := [v]^-$, $\bar{l}_u^i[v]^- := [[u]^- , \bar{l}_u^{i-1}[v]^-]$, $\bar{r}_u^0[v]^- := [v]^-$, $\bar{r}_u^i[v]^- := [\bar{r}_u^{i-1}[v]^- , [u]^-]$, $i \geq 1$ for any $u, v \in \mathfrak{B}(V)$. Let $(k)_a := 1 + a + a^2 + \dots + a^{k-1}$; $(k)_a! := (1)_a(2)_a \dots (k)_a$.

Recall the dual $\mathfrak{B}(V^*)$ of Nichols algebra $\mathfrak{B}(V)$ of rank n in Section 1.3 of [6] and [8]. Let y_1, y_2, \dots, y_n be a dual basis of x_1, x_2, \dots, x_n . $\delta(y_i) = g_i^{-1} \otimes y_i$, $g_i \cdot y_j = p_{ij}^{-1} y_j$ and $\Delta(y_i) = g_i^{-1} \otimes y_i + y_i \otimes 1$. There exists a bilinear map \langle, \rangle from $(\mathfrak{B}(V^*) \# kG) \times \mathfrak{B}(V)$ to $\mathfrak{B}(V)$ such that $\langle y_i, uv \rangle = \langle y_i, u \rangle v + g_i^{-1} \cdot u \langle y_i, v \rangle$ and $\langle y_i, \langle y_j, u \rangle \rangle = \langle y_i y_j, u \rangle$ for any $u, v \in \mathfrak{B}(V)$. Furthermore, for any $u \in \bigoplus_{i=1}^{\infty} \mathfrak{B}(V)_{(i)}$, one has that $u = 0$ if and only if $\langle y_i, u \rangle = 0$ for any $1 \leq i \leq n$. Let i denote x_i in short, sometimes.

Lemma 4.1. *Let $a := p_{ii}^{-1}$, $b := p_{ij}^{-1}$, $c := p_{ji}^{-1}$. Then*

$$(i) \langle y_j, [x_i, x_j]^- \rangle = (p_{ji}^{-1} - 1)x_i, \langle y_i, [x_i, x_j]^- \rangle = (1 - p_{ij}^{-1})x_j,$$

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- (ii) (See Lemma 1.3.3 in [6]) $\langle y_i, x_i^m \rangle = (m)_a x_i^{m-1}$.
- (iii) $\bar{l}_i^m[j]^- = \sum_{k=0}^m (-1)^k \binom{m}{k} x_i^{m-k} x_j x_i^k$.
- (iv) $\langle y_i, \bar{l}_i^m[j]^- \rangle = \sum_{k=0}^{m-1} (-1)^k \left\{ \binom{m}{k} (m-k)_a - \binom{m}{k+1} ((k+1)_a a^{m-k-1} b) \right\} x_i^{m-k-1} x_j x_i^k$.
- (v) $\langle y_i^m y_j, \bar{l}_i^m[j]^- \rangle = (c-1)^m (m)_a!$.
- (vi) $\langle y_i^{m-1} y_j y_i, \bar{l}_i^m[j]^- \rangle = \frac{(m-1)_a!}{c(1-a)} ((c-1)^m (1-ba^m c) + (ac-1)^m (cb-1))$, when $a \neq 1$
- (vii) $\bar{l}_i^m[j]^- \neq 0$, when $c \neq 1$ and $\text{ord}(a) > m$.
- (viii) $\bar{l}_i^m[j]^- \neq 0$, when $\bar{l}_i^m[j] \neq 0$ and $a \neq 1$.
- (ix) $x_i^m \notin \mathfrak{L}^-(V)$ for $1 < m$ and $x_i^m \neq 0$.
- (x) $\dim(\mathfrak{L}^-(V)) = \infty$ when there exists i and j with $i \neq j$, $p_{ji} \neq 1$ and $\text{ord}(p_{ii}) = 1$ or ∞ .

Proof. (i) It can be proved by simple computation.

(iii) and (iv) They can be proved by induction.

(v) It follows from Part (ii) and Part (iii).

(vi)

$$\begin{aligned}
 & \langle y_i^{m-1} y_j y_i, \bar{l}_i^m[j]^- \rangle \\
 &= \sum_{k=0}^{m-1} (-1)^k \left\{ \binom{m}{k} (a^{m-k} - 1) - \binom{m}{k+1} (a^{k+1} - 1) a^{m-k-1} b \right\} c^{m-k-1} (a-1)^{-m+1} \prod_{t=2}^{m-1} (a^t - 1) \\
 &= \frac{(m-1)_a!}{c(1-a)} ((c-1)^m (1-ba^m c) + (ac-1)^m (cb-1)).
 \end{aligned}$$

(vii) It follows from Part (v).

(viii) It follows from Part (v) and Part (vi).

(ix) We can show this by induction on m .

(x) It follows from (v). \square

We use $[x_{i_1} \cdots x_{i_m}]$ instead of $[x_{i_1} [x_{i_2} [x_{i_3} \cdots x_{i_m}]] \cdots]$, $1 < m \leq n$. Similarly we define $[x_{i_1} \cdots x_{i_m}]^- := [x_{i_1} [x_{i_2} [x_{i_3} \cdots x_{i_m}]^-]^- \cdots]^-$.

Lemma 4.2. If $\langle y_i, v \rangle = 0$ and v is a homogeneous element, then

- (i) $\langle y_i, [x_i, v] \rangle = (p_{iv}^{-1} - p_{vi})v$, $\langle y_i y_j, [x_i, v] \rangle = (p_{iv}^{-1} p_{ij} - p_{vi} p_{ji}^{-1}) \langle y_j, v \rangle$, $j \neq i$;
- (ii) $\langle y_i, [x_i, v]^- \rangle = (1 - p_{iv}^{-1})v$, $\langle y_i y_j, [x_i, v]^- \rangle = (p_{ji}^{-1} - p_{iv}^{-1} p_{ij}) \langle y_j, v \rangle$, $j \neq i$.

Lemma 4.3. Assume $p_{i_s, i_t} \begin{cases} \neq 1, & t = s+1 \\ = 1, & s+1 < t \end{cases}$. If i_1, i_2, \dots, i_m is different each

other and $u_{s_k}^- := [x_{i_s}, x_{i_{s+1}}, \dots, x_{i_k}]^-$, then

- (i) $u_{s_k}^- \neq 0$ for any $1 \leq s < k \leq m$.

(ii) $[u_{st}^-, u_{s+1,t}^-]^- \neq 0$ for $1 < s+1 < t \leq m$ and $p_{i_k, i_k} \neq -1$ for $s+1 \leq k \leq t$.

Proof. By Lemma 4.2(ii), $\langle y_{i_s}, u_{sk}^- \rangle = (1 - p_{i_s, i_{s+1}}^{-1})u_{s+1,k}^-$. We can show that $u_{sk}^- \neq 0$ by induction on $|k - s|$. It is clear that $\langle y_{i_s}, [u_{st}^-, u_{s+1,t}^-]^- \rangle = (1 - p_{i_s, i_{s+1}}^{-1})^2(u_{s+1,t}^-)^2$, $\langle y_{i_s}^2, (u_{st}^-)^2 \rangle = (1 + p_{i_s, i_s}^{-1})p_{i_s, i_{s+1}}^{-1}(1 - p_{i_s, i_{s+1}}^{-1})^2u_{s+1,t}^-$ and $\langle y_{i_s}^2, (u_{s,s+1}^-)^2 \rangle = (1 + p_{i_s, i_s}^{-1})p_{i_s, i_{s+1}}^{-1}(1 - p_{i_s, i_{s+1}}^{-1})^2x_{s+1}^2$. Consequently, $\langle y_{i_{t-1}}^2 \cdots y_{i_{s+1}}^2 y_{i_s}, [u_{st}^-, u_{s+1,t}^-]^- \rangle = \{(1 + p_{i_{s+1}, i_{s+1}}^{-1})(1 + p_{i_{s+2}, i_{s+2}}^{-1}) \cdots (1 + p_{i_{t-1}, i_{t-1}}^{-1})\} \{p_{i_{s+1}, i_{s+2}}^{-1} \cdots p_{i_{t-1}, i_t}^{-1}\} \{(1 - p_{i_s, i_{s+1}}^{-1})^2(1 - p_{i_{s+1}, i_{s+2}}^{-1})^2 \cdots (1 - p_{i_{t-1}, i_t}^{-1})^2\} x_{i_t}^2$. \square

Lemma 4.4. (i) Assume $p_{i_s, i_t} \begin{cases} \neq 1, & t = s+1 \\ = 1, & s+1 < t \end{cases}$. If i_1, i_2, \dots, i_m is different each other, then

$$\dim \mathfrak{L}^-(V) \geq \sum_{k=2}^m (N_{i_k} - 2) + C_m^2 + n + |C|, \quad (4.1)$$

where $C := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid p_{i_k, i_k} \neq -1, s+1 \leq k \leq t\}$ and $N_{i_k} := \text{ord}(p_{i_k, i_k})$ when $p_{i_k, i_k} \neq 1$; $N_{i_k} := \infty$ when $p_{i_k, i_k} = 1$.

(ii) Assume $p_{i_s, i_{s+1}} \neq 1, p_{i_{s+1}, i_s} \neq 1, p_{i_s, i_t} = 1$, for any $s+1 < t \leq m$ and $1 \leq s < m$. If i_1, i_2, \dots, i_m is different each other then

$$\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{m-1} (N_{i_k} - 2) + \sum_{k=2}^m (N_{i_k} - 2) + C_m^2 + n + |C|, \quad (4.2)$$

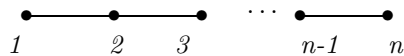
where $C := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid p_{i_k, i_k} \neq -1, s+1 \leq k \leq t\} \cup \{[u_{s,t}^-, u_{s,t+1}^-]^- \mid 1 \leq s < t < m, p_{i_k, i_k} \neq -1, s \leq k \leq t\}$ and $N_{i_k} := \text{ord}(p_{i_k, i_k})$ when $p_{i_k, i_k} \neq 1$; $N_{i_k} := \infty$ when $p_{i_k, i_k} = 1$.

Proof. (i) Let $A := \{\bar{l}_{i_{s+1}}^k(i_s)^- \mid 2 \leq k < N_{i_{s+1}}, 1 \leq s < m\}$, $B := \{u_{s,t}^- \mid 1 \leq s < t \leq m\}$; $E := \{x_j \mid 1 \leq j \leq n\}$. By Lemma 4.1 and Lemma 4.3, $A \cup B \cup C \cup E$ is linearly independent. It is clear that $|A| = \sum_{k=2}^m (N_{i_k} - 2)$; $|B| = C_m^2$. Thus $\dim \mathfrak{L}^-(V) \geq |A \cup B \cup C \cup E|$.

(ii) Let $A := \{\bar{l}_{i_t}^k(i_s)^- \mid 2 \leq k < N_{i_t}, |t - s| = 1, 1 \leq s, t \leq m\}$, $B := \{u_{s,t}^- \mid 1 \leq s < t \leq m\}$; $E := \{x_j \mid 1 \leq j \leq n\}$. By Lemma 4.1 and Lemma 4.3, $A \cup B \cup C \cup E$ is linearly independent. It is clear that $|A| = \sum_{k=1}^{m-1} (N_{i_k} - 2) + \sum_{k=2}^m (N_{i_k} - 2)$; $|B| = C_m^2$. Thus $\dim \mathfrak{L}^-(V) \geq |A \cup B \cup C \cup E|$. \square

Theorem 4.5. Assume that V is a connected finite Cartan type with Cartan matrix $(a_{ij})_{n \times n}$. Let $N := \text{ord}(q_{11})$ and $1 \neq q \in F$; $q_{ij} = q_{ji}$, $1 \leq i, j \leq n$.

(i) $A_n, n > 1$:



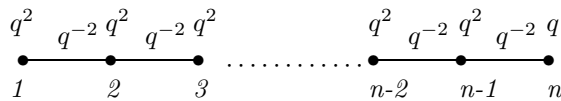
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where $q_{ij}q_{ji} = q^{-1}$, $q_{ii} = q$ for $1 \leq i, j \leq n$.

Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq C_n^2 + n$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq 2(n-1)(N-2) + 3C_n^2 - 2(n-1) + n = 2(n-1)(N-2) + 3C_n^2 - n + 2$.

(ii) B_n , $n > 2$:

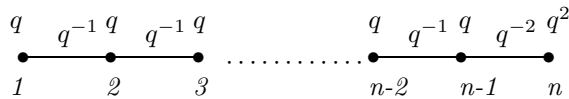


$N_i = N = \text{ord}(q^2)$, $N_n = \text{ord}(q)$, $1 \leq i \leq n-1$.

Case 1. $q^4 = 1$ and $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq (N_n - 2) + C_n^2 + n = N_n - 2 + C_n^2 + n$.

Case 2. $q^4 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{n-1} (N_k - 2) + \sum_{k=2}^n (N_k - 2) + 3C_n^2 - 2(n-1) + n = 2(n-1)(N-2) + (N_n - 2) - (N-2) + 3C_n^2 - n + 2$.

(iii) C_n , $n > 2$:

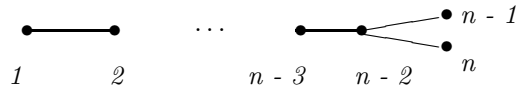


$N_i = N = \text{ord}(q)$, $N_n = \text{ord}(q^2)$, $1 \leq i \leq n-1$.

Case 1. $q^4 = 1$ and $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{n-2} (N_k - 2) + \sum_{k=2}^{n-1} (N_k - 2) + 3C_{n-1}^2 - 2(n-2) + n - 1 + \{N_{n-1} - 2 + (n-1) + (n-2) + 1\} = 2(n-2)(N-2) + (N-2) + 3C_{n-1}^2 + n - 1$.

Case 2. $q^4 \neq 1$. $\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{n-1} (N_k - 2) + \sum_{k=2}^n (N_k - 2) + 3C_n^2 - 2(n-1) + n = 2(n-1)(N-2) + (N_n - 2) - (N-2) + 3C_n^2 - n + 2$.

(iv) D_n ($n > 3$): generalized Dynkin diagram:

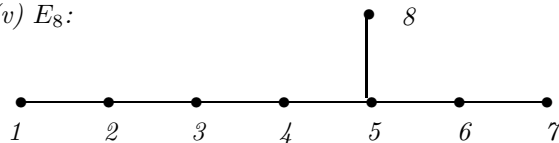


where $q_{ij}q_{ji} = q^{-1}$, $q_{ii} = q$ for $1 \leq i, j \leq n$.

Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq C_{n-1}^2 + 2n - 1$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq (2(n-1-1)(N-2) + 3C_{n-1}^2 - 2(n-1-1) + n - 1) + (2(N-2) + (n-1) + (n-3) + (n-3) + 2 + 1) = 2(n-2)(N-2) + 3C_{n-1}^2 + 2n + 2N - 5$.

(v) E_8 :



where $q_{ij}q_{ji} = q^{-1}$, $q_{ii} = q$ for $1 \leq i, j \leq n$.

Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq C_{8-1}^2 + 8 + 7 = C_7^2 + 15$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq (2(7-1)(N-2) + 3C_7^2 - 2(7-1) + 7) + (2(N-2) + 7 + 4 + 4 + 4 + 1) = 14N + 50$.

(vi) E_7 . Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq C_{7-1}^2 + 7 + 6 = C_6^2 + 13$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq (2(6-1)(N-2) + 3C_6^2 - 2(6-1) + 6) + (2(N-2) + 6 + 3 + 3 + 4 + 1) = 12N + 34$.

(vii) E_6 . Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq C_{6-1}^2 + 6 + 5$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq (2(5-1)(N-2) + 3C_5^2 - 2(5-1) + 5) + (2(N-2) + 5 + 2 + 2 + 4 + 1) = 10N + 21$.

(viii) F_4 .

$$\begin{array}{ccccccc} & q^2 & & q^{-2} & q^2 & & q^{-2} & q & & q^{-1} & q \\ & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\ 1 & & & 2 & & 3 & & 4 & & & \end{array}$$

Case 1. $q^4 = 1$ and $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq 2(N_3 - 2) + (N_4 - 2) + C_4^2 + 4 + 4$.

Case 2. $q^4 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{4-1} (N_k - 2) + \sum_{k=2}^4 (N_k - 2) + 3C_4^2 - 2(4-1) + 4$.

(ix) G_2 :

$$\begin{array}{ccc} & q & \\ & \bullet & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} & q^{-3} & q^3 \\ & \bullet & \\ & & \end{array}$$

Case 1. $q^2 = 1$. Then $\dim \mathfrak{L}^-(V) \geq 1 + 2$.

Case 2. $q^2 \neq 1$. Then $\dim \mathfrak{L}^-(V) \geq \sum_{k=1}^{2-1} (N_k - 2) + \sum_{k=2}^2 (N_k - 2) + 3$.

Proof. (i) $N_i = N = \text{ord}(q)$. Case 1. Let $A := \emptyset$; $B := \{u_{s,t}^- \mid 1 \leq s < t \leq n\}$; $C := \emptyset$.

Case 2. Let $A := \{\bar{l}_t^k(s)^- \mid 2 \leq k < N_t, |t-s|=1, 1 \leq s, t \leq n\}$, $B := \{u_{s,t}^- \mid 1 \leq s < t \leq n\}$; $C := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid 1 < s+1 < t \leq n\} \cup \{[u_{s,t}^-, u_{s,t+1}^-]^- \mid 1 \leq s < t < n\}$; $E := \{x_j \mid 1 \leq j \leq n\}$. By Lemma 4.1 and Lemma 4.3, $A \cup B \cup C \cup E$ is linearly independent. Thus $\dim \mathfrak{L}^-(V) \geq |A \cup B \cup C \cup E|$.

(ii) $N_i = N = \text{ord}(q^2)$, $N_n = \text{ord}(q)$, $1 \leq i \leq n-1$.

Case 1. Let $A := \{\bar{l}_n^k(n-1)^- \mid 2 \leq k < N_n\}$, $B := \{u_{s,t}^- \mid 1 \leq s < t \leq n\}$; $C := \emptyset$; $E := \{x_j \mid 1 \leq j \leq n\}$. Consequently, $\dim \mathfrak{L}^-(V) \geq (N_n - 2) + C_n^2 + n$.

Case 2. It follows from Lemma 4.4 (ii) and the proof of Part (i).

(iii) $N_i = N = \text{ord}(q)$, $N_n = \text{ord}(q^2)$, $1 \leq i \leq n-1$.

Case 1. Let $A := \{\bar{l}_t^k(s)^- \mid 2 \leq k < N_t, |t-s|=1, 1 \leq s, t \leq n-1\} \cup \{\bar{l}_{n-1}^k(n)^- \mid$

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$2 \leq k < N_{n-1}$, $B := \{u_{s,t}^- \mid 1 \leq s < t \leq n\}$; $C := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid 1 < s+1 < t < n\} \cup \{[u_{s,t}^-, u_{s,t+1}^-]^- \mid 1 \leq s < t < n, \}$; $E := \{x_j \mid 1 \leq j \leq n\}$. By Lemma 4.1 and Lemma 4.3, $A \cup B \cup C \cup E$ is linearly independent. Thus $\dim \mathfrak{L}^-(V) \geq |A \cup B \cup C \cup E| = \sum_{k=1}^{n-2} (N_k - 2) + \sum_{k=2}^{n-1} (N_k - 2) + 2C_{n-1}^2 - (n-3) + n-1 + \{N_{n-1} - 2 + (n-1) + (n-2) + 1\}$.
Case 2. It follows from Lemma 4.4 (ii) and the proof of Part (i).

(iv) Let $A_1 := \{\bar{l}_s^k(t)^- \mid 2 \leq k < N, |t-s|=1, 1 \leq s; t \leq n-1\}$, $B_1 := \{u_{s,t}^- \mid 1 \leq s < t \leq n-1\}$; $C_1 := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid 1 < s+1 < t \leq n-1\} \cup \{[u_{s,t}^-, u_{s,t+1}^-]^- \mid 1 \leq s < t < n-1\}$; $E_1 := \{x_j \mid 1 \leq j \leq n-1\}$; $A_2 := \{\bar{l}_s^k(t)^- \mid s = n-2, t = n, \text{ or } s = n, t = n-2\}$, $B_2 := \{[x_s, x_{s+1}, \dots, x_{n-2}, x_n]^- , [x_{n-1}, x_{n-2}, x_n]^- \mid 1 \leq s \leq n-2\}$; $C_2 := \{[[x_s, x_{s+1}, \dots, x_{n-2}, x_n]^- , [x_{s+1}, x_{s+2}, \dots, x_{n-2}, x_n]^-]^- \mid 1 \leq s < n-2\} \cup \{[[x_s, x_{s+1}, \dots, x_{n-2}, x_n]^- , [x_s, x_{s+1}, \dots, x_{n-2}]^-]^- \mid 1 \leq s < n-2\} \cup \{[[x_{n-1}, x_{n-2}, x_n]^- , [x_{n-2}, x_n]^-]^- \} \cup \{[[x_{n-1}, x_{n-2}, x_n]^- , [x_{n-1}, x_{n-2}]^-]^- \}$; $E_2 := \{x_n\}$.

(v) Let $A_1 := \{\bar{l}_s^k(t)^- \mid 2 \leq k < N, |t-s|=1, 1 \leq s; t \leq 7\}$, $B_1 := \{u_{s,t}^- \mid 1 \leq s < t \leq 7\}$; $C_1 := \{[u_{s,t}^-, u_{s+1,t}^-]^- \mid 1 < s+1 < t \leq 7\} \cup \{[u_{s,t}^-, u_{s,t+1}^-]^- \mid 1 \leq s < t < 7\}$; $E_1 := \{x_j \mid 1 \leq j \leq 7\}$; $A_2 := \{\bar{l}_s^k(t)^- \mid s = 5, t = 8, \text{ or } s = 8, t = 5\}$, $B_2 := \{[x_s, x_{s+1}, \dots, x_5, x_8]^- , [x_7, x_6, x_5, x_8]^- , [x_6, x_5, x_8]^- \mid 1 \leq s \leq 5\}$; $C_2 := \{[[x_s, x_{s+1}, \dots, x_5, x_8]^- , [x_{s+1}, x_{s+2}, \dots, x_5, x_8]^-]^- \mid 1 \leq s < 5\} \cup \{[[x_s, x_{s+1}, \dots, x_5, x_8]^- , [x_s, x_{s+1}, \dots, x_5]^-]^- \mid 1 \leq s < 5\} \cup \{[[x_7, x_6, x_5, x_8]^- , [x_6, x_5, x_8]^-]^- ; [[x_6, x_5, x_8]^- , [x_5, x_8]^-]^- ; [[x_7, x_6, x_5, x_8]^- , [x_7, x_6, x_5]^-]^- ; [[x_6, x_5, x_8]^- , [x_6, x_5]^-]^- \}$; $E_2 := \{x_j \mid 8 \leq j \leq 8\}$.

(vi) and (vii) They are similar to the proof of (v).

(viii) Case 1. Let $A := \{\bar{l}_4^k(3)^-, \bar{l}_3^j(4)^-, \bar{l}_3^j(2)^- \mid 2 \leq k < N_4, 2 \leq j < N_3\}$; $B := \{u_{s,t}^- \mid 1 \leq s < t \leq 4\}$; $C := \{[u_{1,4}^-, u_{1+1,4}^-]^- ; [u_{2,4}^-, u_{2+1,4}^-]^- ; [u_{1,4}^-, u_{1,3}^-]^- ; [u_{2,4}^-, u_{2,3}^-]^- \}$; $E := \{x_j \mid 1 \leq j \leq 4\}$. Consequently, $\dim \mathfrak{L}^-(V) \geq 2(N_2 - 2) + (N_4 - 2) + C_4^2 + 4 + 4$.

Case 2. It follows from Lemma 4.4 (ii) and the proof of Part (i).

(ix) Case 1. Let $A := \emptyset$; $B := \{u_{1,2}^-\}$; $E := \{x_j \mid 1 \leq j \leq 2\}$. Consequently, $\dim \mathfrak{L}^-(V) \geq 1 + 2$.

Case 2. It follows from Lemma 4.4 (ii) and the proof of Part (i). \square

We remark that it is possible that the dimensions of Nichols Lie algebras of two twisting equivalent braided vector spaces are different.

Example 4.6. Let V, V' and V'' are three braided vector spaces with braiding matrixes $(q_{ij})_{2 \times 2}$, $(q'_{ij})_{2 \times 2}$ and $(q''_{ij})_{2 \times 2}$, respectively. Assume $q_{1,2}q_{2,1} = 1$, $q_{1,1} =$

$q_{2,2} = -1$, $q_{ii} = q'_{ii} = q''_{ii}$, $q_{ij}q_{ji} = q'_{ij}q'_{ji} = q''_{ij}q''_{ji}$ for $i, j = 1, 2$. Then

- (i) $\dim \mathfrak{B}(V) = 4$.
- (ii) $[x_1, x_2]^- = 0$ and $\dim \mathfrak{L}^-(V') = 2$ when $q'_{1,2} = 1$.
- (iii) $[x_1, x_2]^- \neq 0$ and $\dim \mathfrak{L}^-(V'') = 3$ when $q''_{1,2} \neq 1$.
- (iv) V' and V'' are twisting equivalent.

Lemma 4.7. (i) If there exists $1 \leq i \leq n$ such that $\text{ord}(p_{ii}) = 1$ or ∞ , then there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$.

(ii) If $p_{11} = p_{22} = -1$, then $\langle (y_2y_1)^k, x_1(x_2x_1)^m \rangle = \prod_{j=0}^{k-1} (1 - (p_{12}p_{21})^{-m-j})x_1(x_2x_1)^{m-k}$ for $1 \leq k \leq m$.

(iii) If $p_{11} = p_{22} = -1$ and $\text{ord}(p_{12}p_{21}) = \infty$, then $\bar{l}_{[x_2, x_1]^-}^m(x_1)^- \neq 0$ for any $m \in \mathbb{N}$, which implies $\dim(\mathfrak{L}^-(V)) = \infty$.

(iv) If the generalized Dynkin diagram of V is a simple chain with length d and there exists $1 \leq i < d$ such that $\text{ord}(p_{i,i+1}p_{i+1,i}) = \infty$, then there exists $1 \leq j \leq n$ such that $\text{ord}(p_{jj}) = \infty$ or $\dim(\mathfrak{L}^-(V)) = \infty$.

(v) If $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there exists $u \in D(V)$ such that $\text{ord}(p_{uu}) = \infty$, then there exists $1 \leq i \leq n$ such that $\text{ord}(p_{ii}) = \infty$ or $\dim(\mathfrak{L}^-(V)) = \infty$.

Proof. (i) It follows from Lemma 4.1 (x).

(ii) It can be proved by induction on m that $\langle y_1, x_1(x_2x_1)^m \rangle = (x_2x_1)^m + p_{1,12}^{-m}(x_1x_2)^m$ and $\langle y_2y_1, x_1(x_2x_1)^m \rangle = (1 - (p_{12}p_{21})^{-m})x_1(x_2x_1)^{m-1}$. It can be proved by induction on k that $\langle (y_2y_1)^k, x_1(x_2x_1)^m \rangle = \prod_{j=0}^{k-1} (1 - (p_{12}p_{21})^{-m-j})x_1(x_2x_1)^{m-k}$.

(iii) It follows from (ii).

(iv) If $1 \leq i < d$ and $q_{ii} \neq -1$, then $\text{ord}(q_{ii}) = \infty$ by Definition 1 in [7]. Similarly, $q_{i+1,i+1} \neq -1$, then $\text{ord}(q_{i+1,i+1}) = \infty$. If $q_{ii} = -1$ and $q_{i+1,i+1} = -1$, then we complete the proof by Part (ii).

(v) If there exists $u \in D(V)$ such that $\text{ord}(p_{uu}) = \infty$, then there exists a $1 \leq i \leq n$ such that $\text{ord}(p_{ii}) = \infty$ except the cases of Row 3, Diagram 2, Table 1; Row 8, Diagram 2, Table 2; Row 9, Diagram 4, Table 2; Row 10, Diagram 3, Table 2, in [6], and the cases of Row 10, Diagram 6, Appendix B; Row 12, Diagram 5, Appendix B. Row 1-10, Appendix C, in [7]. By (iii), $\dim(\mathfrak{L}^-(V)) = \infty$. \square

Theorem 4.8. Assume that $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and there does not exist any m -infinity element with $p_{u,u} \neq 1$ for any $u \in D(V)$. Then

(i) $\dim(\mathfrak{B}(V)) = \infty$ if and only if there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$.

(ii) $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}^-(V)$ is finite-dimensional when the input of every vertex in the directed Dynkin diagram of V is more than 0, i.e. for any $1 \leq i \leq n$, there exists j such that $p_{ji} \neq 1$.

Proof. By [6] $\mathfrak{B}(V)$ is finite-dimensional if and only if $1 < \text{ord}(p_{uu}) < \infty$ for any $u \in D(V)$.

(i) The sufficiency is clear. The necessity. By [6] there exists a $u \in D(V)$ such that $\text{ord}(p_{uu}) = 1$ or ∞ . We complete the proof by Lemma 4.7 (i), (iv), (v).

(ii) It follows from Lemma 4.1(x) and the proof of Part (i). \square

Corollary 4.9. *Assume that $\Delta(\mathfrak{B}(V))$ is an arithmetic root system. Then $\dim(\mathfrak{B}(V)) = \infty$ if and only if there exists V' , which is twisting equivalent to V , such that $\dim(\mathfrak{L}^-(V')) = \infty$ in the following three cases:*

(i) $\dim V = 2$.

(ii) V is of finite Cartan type.

(iii) V is a Yetter-Drinfeld module over finite cyclic groups.

Proof. It is enough to show that there does not exist any m -infinity element and $p_{u,u} \neq 1$ for any $u \in D(V)$ in this three cases.

There are no m -infinity elements in this three cases by Theorem 3.10 and Lemma 6.3. The second claim is obtained by Appendix, Lemma 6.4 in [16] and Theorem 3.3 in [15]. \square

Theorem 4.10. *If V is a quantum linear space and $0 \neq |p_{s,t}| < 1$ when $t = s + 1$; $p_{st} = 1$ when $t \neq s + 1$ and $s < t$, then $\dim \mathfrak{L}^-(V) = N_1 \cdots N_n - (N_1 + \cdots + N_n - n + 1)$ and $\mathfrak{B}(V) = \mathfrak{L}^-(V) \oplus \text{span}\{x_i^{m_i} \mid 1 < m_i \in \mathbb{N}\} \oplus F$.*

Proof. We show this by following several steps.

Let $W = x_1^{a_1} \cdots x_n^{a_n}$ and $N(W) := a_1 + \cdots + a_n$. Assume a_{i_0} is the first non-zero element and a_{j_0} is the last non-zero element.

(i) If $u = x_1^{a_1} \cdots x_r^{a_r}$ and $v = x_{r+1}^{a_{r+1}} \cdots x_n^{a_n}$, then $uv = p_{u,v}vu \in \mathfrak{L}^-(V)$ and $p_{u,v} = \prod_{i=1, \dots, r; j=r+1, \dots, n} p_{i,j}^{a_i a_j}$ when $u, v \in \mathfrak{L}^-(V)$.

(ii) We use induction on $N(W)$ to show that $W \in \mathfrak{L}^-(V)$ when $a_i \leq 1$ for $1 \leq i \leq n$. It is clear for $N(W) = 1$. For $N(W) > 1$, Let $1 \leq i_0 \leq n$ such that a_{i_0} is the first non-zero element. Let $u = x_{i_0}$, $v = x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$. By induction we have $v \in \mathfrak{L}^-(V)$. Consequently, $W = uv \in \mathfrak{L}^-(V)$ by (i).

(iii) We use induction on $N(W)$ to show that $W \in \mathfrak{L}^-(V)$ when there exist two non-zero elements in $\{a_1, a_2, \dots, a_n\}$. When $N(W) = 2$, this is Case (i). For $N(W) = 3$, considering (i) we can assume that there exist only two $a_{i_0} \neq 0$ and $a_{j_0} \neq 0$. If $a_{i_0} > 1$, then $a_{j_0} > 1$. Let $u = x_{i_0}$, $v = x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$. By induction, $v \in \mathfrak{L}^-(V)$. Consequently, $W \in \mathfrak{L}^-(V)$. If $a_{i_0} = 1$, let $u = x_{i_0}^{a_{i_0}-1} \cdots x_{j_0}^{a_{j_0}-1}$, $v = x_{j_0}$. By induction, $u \in \mathfrak{L}^-(V)$. Consequently, $W = uv \in \mathfrak{L}^-(V)$. For $N(W) > 3$, if $a_{i_0} = 1$ and $a_{j_0} = 1$, let $u = x_{i_0}$, $v = x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$. By induction, $v \in \mathfrak{L}^-(V)$. Consequently, $W = uv \in \mathfrak{L}^-(V)$. If $a_{i_0} > 1$, let $u = x_{i_0}$, $v = x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$. By induction, $v \in \mathfrak{L}^-(V)$. Consequently, $W \in \mathfrak{L}^-(V)$. If $a_{i_0} = 1$ and $a_{j_0} > 1$, let $u = x_{i_0}^{a_{i_0}-1} \cdots x_{j_0}^{a_{j_0}-1}$, $v = x_{j_0}$. By induction, $u \in \mathfrak{L}^-(V)$. Consequently, $W = uv \in \mathfrak{L}^-(V)$.

(iv) By Theorem 2 in [11] or Theorem 10 in [9], $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i < N_i, 1 \leq i \leq n\}$ is a basis of $\mathfrak{B}(V)$. Considering (i) - (iii) and Lemma 4.1 (ix) we complete the proof. \square

Let us remark that it is possible that $\dim \mathfrak{B}(V) = \infty$ and $\dim \mathfrak{L}^-(V) < \infty$.

Example 4.11. Assume $q_{12} = q_{21} = 1$ and $q_{11} = q_{22} = 1$. Then $\dim \mathfrak{B}(V) = \infty$ and $\dim \mathfrak{L}^-(V) = 2$.

5. Two examples of Lie algebras without maximal solvable ideals

In this section we find two examples of Lie algebras which have no maximal solvable ideals.

Let $R_k := \{x \mid x \text{ is a primitive } k\text{-th unit root}\}$. Assume $(L, [\]^-)$ is a Lie algebra. Let $D^{(0)}(L) := L$, $D^{(1)}(L) = [L, L]^-$, $D^{(k+1)}(L) = [D^{(k)}(L), D^{(k)}(L)]^-$. If there exists a natural number m such that $D^{(m)}(L) = 0$, then L is called a solvable Lie algebra.

For any $z_1, z_2, \dots, z_k \in L$, let $\sigma(z_1, z_2) := [z_1, z_2]^-$, $\sigma(z_1, z_2, z_3, z_4) := [\sigma(z_1, z_2), \sigma(z_3, z_4)]^-$. \dots , $\sigma(z_1, z_2, \dots, z_{2^k}) := [\sigma(z_1, z_2, \dots, z_{2^{k-1}}), \sigma(z_{2^{k-1}+1}, \dots, z_{2^k})]^-$.

Lemma 5.1. Assume that V_m is a quantum vector space with $q_{i,i+1} \neq 1$ and $q_{ij} = 1$, $i+1 < j \leq m$, $1 \leq i < m$. If x_1, x_2, \dots, x_m be the canonical basis of V_m and $k-s$ is a power of 2, then $\langle y_s, \sigma(x_s, \dots, x_{k-1}) \rangle \neq 0$ for any $1 \leq s < k \leq m+1$.

Proof. Now we show that $\langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{k-1}) \rangle \neq 0$ by induction on $k-s$, where $k-s$ is a power of 2, for any $1 \leq s < k \leq m+1$. If $k-s = 2$, then $\langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{k-1}) \rangle = (1 - p_{s,s+1}^{-1})x_{s+1} \neq 0$. Assume $k-s > 2$. Then

$$\begin{aligned} & \langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{k-1}) \rangle \\ &= (1 - p_{\frac{k-s}{2}, \frac{k-s}{2}+1}^{-1}) \langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{\frac{k-s}{2}}) \rangle \sigma(x_{\frac{k-s}{2}+1}, x_{\frac{k-s}{2}+2}, \dots, x_{k-1}). \end{aligned}$$

By induction assumption, $\langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{\frac{k-s}{2}}) \rangle \neq 0$ and $\sigma(x_{\frac{k-s}{2}+1}, x_{\frac{k-s}{2}+2}, \dots, x_{k-1}) \neq 0$. Considering the restricted PBW basis of Nichols algebras (see [11, 6]), we have $\langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{\frac{k-s}{2}}) \rangle \sigma(x_{\frac{k-s}{2}+1}, x_{\frac{k-s}{2}+2}, \dots, x_{k-1}) \neq 0$, which implies $\langle y_s, \sigma(x_s, x_{s+1}, \dots, x_{k-1}) \rangle \neq 0$. \square

Example 5.2. (i) Assume that V_m is a quantum vector space with $q_{i,i+1} \neq 1$ and $q_{ij} = 1$, $i+1 < j \leq m$, $1 \leq i < m$. Let $A_k := \bigoplus_{i=1}^k \mathfrak{B}(V_i)$ and $A := \bigoplus_{i=1}^\infty \mathfrak{B}(V_i)$ as associative algebras. Then A^- is not a solvable Lie algebra.

(ii) A^- has no maximal nilpotent ideal and maximal solvable ideal.

Proof. (i) If A^- is solvable, then there exists a natural number m such that $D^{(m)}(A^-) = 0$. Therefore $D^{(m)}(\mathfrak{B}(V_{2^m})^-) = 0$ and $\sigma(x_1, x_2, \dots, x_{2^m}) \in D^{(m)}(\mathfrak{B}(V_{2^m})^-)$, which contradicts to Lemma 5.1.

(ii) It follows from (i). \square

Similarly, we have the following conclusion.

Example 5.3. (i) Assume that V_m is a quantum vector space with $q_{i,i+1} \neq 1$ and $q_{ij} = 1$, $i+1 < j \leq m$, $1 \leq i < m$. Let $L_k := \oplus_{i=1}^k \mathfrak{L}^-(V_i)$ and $L := \oplus_{i=1}^\infty \mathfrak{L}^-(V_i)$ as Lie algebras. Then L is not any solvable Lie algebras.

(ii) L has no maximal nilpotent ideal and maximal solvable ideal.

For a coagebra C , let $C^+ := \{x \in C \mid \epsilon(x) = 0\}$ (see [12, 10, 13]).

Example 5.4. Assume that V_m is a quantum vector space with $q_{i,i+1} \neq 1$ and $q_{ij} = 1$, $i+1 < j \leq m$, $1 \leq i < m$. Let $A_k := \oplus_{i=1}^k \mathfrak{B}(V_i)$ and $A := \oplus_{i=1}^\infty \mathfrak{B}(V_i)$ as associative algebras. Then

(i) A^+ is not a nilpotent ideal of associative algebra A .

(ii) A has no maximal nilpotent ideal.

(iii) A^+ is a Baer radical of A , where Baer ideal is defined in [14].

Proof. (i) It follows Lemma 5.1.

(ii) If A has the maximal ideal I , then $I = A^+$ since every $\mathfrak{B}(V_i)^+$ is nilpotent for any $i \in \mathbb{N}$. This contradicts to (i).

(iii) $A^+ = \sum_{k=1}^\infty (A_k)^+$ and $(A_1)^+ \subseteq (A_2)^+ \subseteq \cdots (A_n)^+ \subseteq \cdots$. Since $(A_k)^+$ is nilpotent ideal of A for $k \in \mathbb{N}$, A^+ is Baer radical of A by [14]. \square

6. Appendix

6.1. No m -infinity elements in Nichols algebras over finite cyclic groups

In this subsection of the appendix we show that there does not exist any m -infinity element in Nichols algebra $\mathfrak{B}(V)$ with arithmetic root system $\Delta(\mathfrak{B}(V))$ over finite cyclic groups.

Lemma 6.1. (i) $\langle y_j, u^k \rangle = \sum_{i=0}^{k-1} p_{x_j, u}^{-i} u^i \langle y_j, u \rangle u^{k-i-1}$ for homogeneous element u .

(ii) If $p_{11}^2 = p_{33}^2 = 1$, then

$$\langle y_1, [1, 3]^k \rangle = (-p_{13}^{-1})^k ((p_{13}p_{31})^k - 1) \overbrace{3131 \cdots 1313}^k.$$

(iii) If $p_{33}^2 = p_{22}^2 = 1$ and $[2, 3] = 0$, then

$$\langle y_1, [[1, 3], 2]^k \rangle = \sum_{i=0}^{k-1} (p_{11}p_{12}p_{13}p_{21}p_{31})^{-i} (p_{12}^{-1} - p_{21})(p_{13}^{-1} - p_{31})(p_{21}p_{31})^{k-1} \overbrace{231231 \cdots 312}^k 3.$$

(iv) If $\langle y_i, v \rangle = 0$ and v is a homogeneous element, then $\langle y_i, [v, x_i] \rangle = 0$.

(v) $\langle y_2, [[1, 3], 2]^k \rangle = 0$ and $\langle y_3, [1, 3]^k \rangle = 0$.

Proof. (i) It can be obtained by induction on k .

(ii)

$$\begin{aligned}
 \langle y_1, [1, 3]^k \rangle &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{13}^{-i} [1, 3]^i \langle y_1, [1, 3] \rangle [1, 3]^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{13}^{-i} (31)^i (p_{13}^{-1} - p_{31}) 3 (-p_{31})^{k-i-1} (13)^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{13}^{-i} (-p_{31})^{-i} (p_{13}^{-1} - p_{31}) (31)^i 3 (-p_{31})^{k-1} (13)^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{13}^{-i} p_{31}^{-i} (p_{13}^{-1} - p_{31}) (-p_{31})^{k-1} \overbrace{3131 \cdots 1313}^k \\
 &= (-p_{13}^{-1})^k ((p_{13} p_{31})^k - 1) \overbrace{3131 \cdots 1313}^k.
 \end{aligned}$$

(iii)

$$\begin{aligned}
 \langle y_1, [[1, 3], 2]^k \rangle &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{12}^{-i} p_{13}^{-i} [[1, 3], 2]^i \langle y_1, [[1, 3], 2] \rangle [[1, 3], 2]^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{12}^{-i} p_{13}^{-i} [[1, 3], 2]^i (p_{12}^{-1} - p_{21}) (p_{13}^{-1} - p_{31}) 23 [[1, 3], 2]^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{12}^{-i} p_{13}^{-i} (231)^i (p_{12}^{-1} - p_{21}) (p_{13}^{-1} - p_{31}) 23 (p_{21} p_{23} p_{31})^{k-i-1} (132)^{k-i-1} \\
 &= \sum_{i=0}^{k-1} p_{11}^{-i} p_{12}^{-i} p_{13}^{-i} (p_{21} p_{31})^{-i} (p_{12}^{-1} - p_{21}) (p_{13}^{-1} - p_{31}) (p_{21} p_{31})^{k-1} (231)^i 23 (123)^{k-i-1} \\
 &= \sum_{i=0}^{k-1} (p_{11} p_{12} p_{13} p_{21} p_{31})^{-i} (p_{12}^{-1} - p_{21}) (p_{13}^{-1} - p_{31}) (p_{21} p_{31})^{k-1} \overbrace{231231 \cdots 312}^k 3.
 \end{aligned}$$

(iv) It is clear.

(v) It follows from (iv). \square

Theorem 6.2. *If $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and V is a YD module over finite cyclic group, then there does not exist any m -infinity element in $\mathfrak{B}(V)$.*

Proof. We show this by several steps. Obviously, it follows that $[x_1, x_2]$, $[x_1, x_3]$ and $[[x_1, x_3], x_2]$ are nilpotent from Lemma 6.1 or simple computation. We only need consider the three cases below by Theorem 2.7 in [15].

(i) Assume generalized Dynkin diagram of braided vector space V is the following condition:

$$\begin{array}{c} -1 \quad q \quad -1 \quad q^{-1} \quad -1 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array}, \quad q \in R_m, m > 2.$$

Then $D := \{[x_1]; [x_2]; [x_3]; [x_1, x_2]; [x_1, x_3]; [[x_1, x_3], x_2]\}$ by Theorem 3.3 (i) in [15]. It follows that every hard super-letter is nilpotent from Lemma 6.1.

(ii) Assume generalized Dynkin diagram of braided vector space V is the following condition:

$$\begin{array}{c} -1 \quad \zeta \quad -1 \quad \zeta \quad -1 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array}, \quad \zeta \in R_3.$$

Then $D = \{[x_1]; [x_2]; [x_3]; [x_1, x_2]; [x_1, x_3]; [[x_1, x_3], x_2]; [x_1, x_2], [x_1, x_3]; [[x_1, x_2], [x_1, x_3], x_2]; [x_1, x_3], [[x_1, x_3], x_2]; [[[x_1, x_2], [x_1, x_3], x_2], [x_1, x_3]]\}$ by Theorem 3.3(i) in [15].

Now we show that every hard super-letter is nilpotent.

$$\begin{aligned} < y_1, [[1, 2], [1, 3]] > = -p_{12}^{-1} p_{13}^{-1} (1 - p_{12} p_{21} p_{31} p_{13}) [2, [1, 3]], \\ < y_1, [[1, 2], [1, 3]]^3 > = \sum_{i=0}^2 p_{11}^{-2i} p_{12}^{-i} p_{13}^{-i} [[1, 2], [1, 3]]^i < y_1, [[1, 2], [1, 3]] > [[1, 2], [1, 3]]^{2-i} \\ &= \sum_{i=0}^2 p_{12}^{-i} p_{13}^{-i} [[1, 2], [1, 3]]^i (-p_{12}^{-1} p_{13}^{-1} (1 - p_{12} p_{21} p_{31} p_{13}) [2, [1, 3]]) [[1, 2], [1, 3]]^{2-i} \\ &= -p_{12}^{-1} p_{13}^{-1} (1 - p_{12} p_{21} p_{31} p_{13}) \{ [2, [1, 3]] [[1, 2], [1, 3]]^2 \\ &\quad + p_{12}^{-1} p_{13}^{-1} [[1, 2], [1, 3]] [2, [1, 3]] [[1, 2], [1, 3]] + p_{12}^{-2} p_{13}^{-2} [[1, 2], [1, 3]]^2 [2, [1, 3]] \} \\ &:= -p_{12}^{-1} p_{13}^{-1} (1 - p_{12} p_{21} p_{31} p_{13}) R, \\ < y_2, [2, [1, 3]] > = p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3], \\ < y_2, [2, [1, 3]]^2 > = p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) \{ [1, 3] [2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3] \}, \\ < y_2, R > = p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) \{ [1, 3] [[1, 2], [1, 3]]^2 \\ &\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-2} p_{22}^{-1} p_{23}^{-1} [[1, 2], [1, 3]] [1, 3] [[1, 2], [1, 3]] + p_{21}^{-4} p_{22}^{-2} p_{23}^{-2} p_{12}^{-2} p_{13}^{-2} [[1, 2], [1, 3]]^2 [1, 3] \} \\ &= p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) \{ p_{11}^4 p_{12}^2 p_{13}^2 p_{31}^4 p_{32}^2 p_{33}^2 [[1, 2], [1, 3]]^2 [1, 3] \\ &\quad + p_{11}^2 p_{12} p_{13} p_{31}^2 p_{32} p_{33} p_{12}^{-1} p_{13}^{-1} p_{21}^{-2} p_{22}^{-1} p_{23}^{-1} [[1, 2], [1, 3]] [[1, 2], [1, 3]] [1, 3] \\ &\quad + p_{21}^{-4} p_{22}^{-2} p_{23}^{-2} p_{12}^{-2} p_{13}^{-2} [[1, 2], [1, 3]]^2 [1, 3] \} \\ &= p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) p_{21}^{-2} p_{32}^2 p_{31}^2 \{ p_{12}^2 p_{21}^2 p_{13}^2 p_{31}^2 + 1 + p_{21}^{-2} p_{13}^{-2} p_{12}^{-2} p_{31}^{-2} \} [[1, 2], [1, 3]]^2 [1, 3] = 0 \\ &\quad \text{since } [[[1, 2], [1, 3]], [1, 3]] = 0, \end{aligned}$$

$$< y_3, R > = 0,$$

$$\begin{aligned} < y_1, R > &= < y_1, < y_1, [[1, 2], [1, 3]]^3 > > = < y_1^2, [[1, 2], [1, 3]]^3 > = 0. \text{ Then } [[1, 2], [1, 3]]^3 = 0, \\ < y_1, [[1, 2], [[1, 3], 2]] > &= -p_{12}^{-1} p_{21} p_{13}^{-1} (1 - \xi)^2 [1, 3] 2, \end{aligned}$$

$$\begin{aligned}
\langle y_1, [[1, 2], [[1, 3], 2]]^2 \rangle &= -p_{12}^{-1} p_{21} p_{13}^{-1} (1 - \xi)^2 2[1, 3] 2[[1, 2], [[1, 3], 2]] \\
&\quad + p_{11}^{-2} p_{12}^{-2} p_{13}^{-1} [[1, 2], [[1, 3], 2]] (-p_{12}^{-1} p_{21} p_{13}^{-1} (1 - \xi)^2 2[1, 3] 2) \\
&= -p_{12}^{-1} p_{21} p_{13}^{-1} (1 - \xi)^2 \cdot \{2[1, 3] 2[[1, 2], [[1, 3], 2]] + p_{11}^{-2} p_{12}^{-2} p_{13}^{-1} [[1, 2], [[1, 3], 2]] 2[1, 3] 2\} \\
&:= -p_{12}^{-1} p_{21} p_{13}^{-1} (1 - \xi)^2 \cdot A.
\end{aligned}$$

It is clear $\langle y_1, A \rangle = 0$, $\langle y_3, A \rangle = 0$, and

$$\begin{aligned}
\langle y_2, A \rangle &= [1, 3] 2[[1, 2], [[1, 3], 2]] - p_{21}^{-1} p_{23}^{-1} 2[1, 3] [[1, 2], [[1, 3], 2]] \\
&\quad + \xi p_{13}^{-1} p_{23}^{-1} [[1, 2], [[1, 3], 2]] [1, 3] 2 - \xi p_{21}^{-1} p_{13}^{-1} p_{23}^{-2} [[1, 2], [[1, 3], 2]] 2[1, 3] \\
&= -p_{21}^{-1} p_{23}^{-1} [[1, 3], 2] [[1, 2], [[1, 3], 2]] - \xi p_{21}^{-1} p_{13}^{-1} p_{23}^{-2} [[1, 2], [[1, 3], 2]] [[1, 3], 2] \\
&= -p_{21}^{-1} p_{23}^{-1} ([[1, 3], 2] [[1, 2], [[1, 3], 2]] + \xi p_{13}^{-1} p_{23}^{-1} [[1, 2], [[1, 3], 2]] [[1, 3], 2]) \\
&= -p_{21}^{-1} p_{23}^{-1} [[[1, 2], [[1, 3], 2]], [[1, 3], 2]] = 0,
\end{aligned}$$

then $[[1, 2], [[1, 3], 2]]^2 = 0$ and $\langle y_1, [[1, 3], [[1, 3], 2]] \rangle = -p_{13}^{-2} p_{21} p_{23} (1 - \xi)^2 3[1, 2] 3$.

We obtain $[[1, 3], [[1, 3], 2]]^2 = 0$ since $[[1, 3], 2] = q_{32}^{-1} [[1, 2], 3]$.

$$\begin{aligned}
\langle y_1, [[[1, 2], [[1, 3], 2]], [1, 3]] \rangle &= p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2, \\
\langle y_1, [[[1, 2], [[1, 3], 2]], [1, 3]]^2 \rangle &= p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 [[[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{11}^{-3} p_{12}^{-2} p_{13}^{-2} [[[1, 2], [[1, 3], 2]], [1, 3]] (p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2) \\
&= p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) \{[2, [1, 3]]^2 [[[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{11}^{-3} p_{12}^{-2} p_{13}^{-2} [[[1, 2], [[1, 3], 2]], [1, 3]] [2, [1, 3]]^2\} \\
&:= p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) B,
\end{aligned}$$

$$\begin{aligned}
\langle y_2, B \rangle &= p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([1, 3] [2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) [[[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{21}^{-3} p_{22}^{-2} p_{23}^{-2} p_{11}^{-3} p_{12}^{-2} p_{13}^{-2} [[[1, 2], [[1, 3], 2]], [1, 3]] p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) \\
&\quad ([1, 3] [2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&:= C,
\end{aligned}$$

$$\begin{aligned}
\langle y_1, C \rangle &= p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ((p_{13}^{-1} - p_{31}) 3[2, [1, 3]] \\
&\quad + p_{11}^{-1} p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] (p_{13}^{-1} - p_{31}) 3) [[[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{12}^{-1} p_{13}^{-2} p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([1, 3] [2, [1, 3]] \\
&\quad + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 \\
&\quad + p_{21}^{-3} p_{22}^{-2} p_{23}^{-2} p_{11}^{-3} p_{12}^{-2} p_{13}^{-2} p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([1, 3] [2, [1, 3]] \\
&\quad + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) + p_{21}^{-3} p_{22}^{-2} p_{23}^{-2} p_{12}^{-4} p_{13}^{-4} [[[1, 2], [[1, 3], 2]], [1, 3]] p_{21}^{-1} p_{23}^{-1} \\
&\quad (1 - p_{12} p_{21}) ((p_{13}^{-1} - p_{31}) 3[2, [1, 3]] + p_{11}^{-1} p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] (p_{13}^{-1} - p_{31}) 3) \\
&= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} (1 - \xi)^2 (3[2, [1, 3]] + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] 3) [[[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{12}^{-3} p_{13}^{-4} p_{21}^{-1} p_{23}^{-1} \xi (1 - \xi)^2 ([1, 3] [2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) [2, [1, 3]]^2 \\
&\quad - p_{13}^{-4} p_{23}^{-3} (1 - \xi)^2 [2, [1, 3]]^2 ([1, 3] [2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad + p_{23}^{-3} p_{12}^{-1} p_{13}^{-5} p_{21}^{-1} (1 - \xi)^2 [[[1, 2], [[1, 3], 2]], [1, 3]] (3[2, [1, 3]] + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] 3) \\
&:= D,
\end{aligned}$$

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$$\begin{aligned}
\langle y_3, D \rangle &= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} (1 - \xi)^2 ([2, [1, 3]] \\
&\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} p_{31}^{-1} p_{32}^{-1} p_{33}^{-1} [2, [1, 3]]) ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad + p_{23}^{-3} p_{12}^{-1} p_{13}^{-5} p_{21}^{-1} (1 - \xi)^2 p_{31}^{-3} p_{32}^{-2} ([[1, 2], [[1, 3], 2]], [1, 3]) ([2, [1, 3]) \\
&\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} p_{31}^{-1} p_{32}^{-1} p_{33}^{-1} [2, [1, 3]]) \\
&= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} (1 - \xi)^3 [2, [1, 3]] ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad + p_{23}^{-1} p_{12}^{-1} p_{13}^{-2} p_{21}^{-1} (1 - \xi)^3 ([[1, 2], [[1, 3], 2]], [1, 3]) [2, [1, 3]) \\
&:= E,
\end{aligned}$$

$$\begin{aligned}
\langle y_1, E \rangle &= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} p_{11}^{-1} p_{12}^{-1} p_{13}^{-1} (1 - \xi)^3 [2, [1, 3]] p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 \\
&\quad + p_{23}^{-1} p_{12}^{-1} p_{13}^{-2} p_{21}^{-1} (1 - \xi)^3 p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 [2, [1, 3]) \\
&= -p_{23}^{-1} p_{13}^{-4} (1 - \xi)^4 p_{12}^{-2} [2, [1, 3]]^3 + p_{23}^{-1} p_{13}^{-4} (1 - \xi)^4 p_{12}^{-2} [2, [1, 3]]^3 = 0,
\end{aligned}$$

$$\begin{aligned}
\langle y_2, E \rangle &= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} (1 - \xi)^3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3] ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad + p_{21}^{-3} p_{23}^{-2} p_{23}^{-1} p_{12}^{-1} p_{13}^{-2} p_{21}^{-1} (1 - \xi)^3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([[1, 2], [[1, 3], 2]], [1, 3]) [1, 3]) \\
&= p_{21}^{-2} p_{23}^{-2} p_{13}^{-1} (1 - \xi)^4 [1, 3] ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad + p_{21}^{-5} p_{23}^{-4} p_{12}^{-1} p_{13}^{-2} (1 - \xi)^4 ([[1, 2], [[1, 3], 2]], [1, 3]) [1, 3]) \\
&= p_{21}^{-2} p_{23}^{-2} p_{13}^{-1} (1 - \xi)^4 \{ [1, 3] ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad + p_{21}^{-3} p_{23}^{-2} p_{12}^{-1} p_{13}^{-1} ([[1, 2], [[1, 3], 2]], [1, 3]) [1, 3]) \} \\
&= 0 \text{ since } [[[1, 2], [[1, 3], 2]], [1, 3], [1, 3]] = 0,
\end{aligned}$$

$$\begin{aligned}
\langle y_2, D \rangle &= p_{21}^{-1} p_{23}^{-1} p_{13}^{-1} (1 - \xi)^2 (p_{23}^{-1} 3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3]) \\
&\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3] 3 ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad - p_{12}^{-3} p_{13}^{-4} p_{21}^{-1} p_{23}^{-1} \xi (1 - \xi)^2 p_{21}^{-2} p_{23}^{-2} ([1, 3] [2, [1, 3]) \\
&\quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([1, 3] [2, [1, 3]) + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad - p_{13}^{-4} p_{23}^{-3} (1 - \xi)^2 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) ([1, 3] [2, [1, 3]) + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad ([1, 3] [2, [1, 3]) - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad + p_{21}^{-3} p_{23}^{-2} p_{23}^{-3} p_{12}^{-1} p_{13}^{-5} p_{21}^{-1} (1 - \xi)^2 ([[1, 2], [[1, 3], 2]], [1, 3]) (p_{23}^{-1} 3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3]) \\
&\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) [1, 3] 3) \\
&= p_{21}^{-2} p_{23}^{-3} p_{13}^{-1} (1 - \xi)^3 (3 [1, 3] + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} [1, 3] 3) ([[1, 2], [[1, 3], 2]], [1, 3]) \\
&\quad - p_{12}^{-3} p_{13}^{-4} p_{21}^{-1} p_{23}^{-4} \xi (1 - \xi)^3 ([1, 3] [2, [1, 3]) \\
&\quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) ([1, 3] [2, [1, 3]) + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad - p_{13}^{-4} p_{23}^{-3} (1 - \xi)^3 p_{21}^{-1} ([1, 3] [2, [1, 3]) + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) ([1, 3] [2, [1, 3]) \\
&\quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]] [1, 3]) \\
&\quad + p_{23}^{-7} p_{12}^{-1} p_{13}^{-5} p_{21}^{-1} (1 - \xi)^3 ([[1, 2], [[1, 3], 2]], [1, 3]) (3 [1, 3] + p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} [1, 3] 3)
\end{aligned}$$

$$\begin{aligned}
&= p_{21}^{-2} p_{23}^{-3} p_{13}^{-1} (1 - \xi)^4 3[1, 3][[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad - p_{13}^{-4} p_{21}^{-1} p_{23}^{-4} \xi (1 - \xi)^3 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3])([1, 3][2, [1, 3]] \\
&\quad \quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad - p_{13}^{-4} p_{23}^{-4} (1 - \xi)^3 p_{21}^{-1} ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3])([1, 3][2, [1, 3]] \\
&\quad \quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad + p_{23}^{-7} p_{12}^{-1} p_{13}^{-5} p_{21}^{-5} (1 - \xi)^4 [[[1, 2], [[1, 3], 2]], [1, 3]] 3[1, 3] \\
&= p_{21}^{-2} p_{23}^{-3} p_{13}^{-1} (1 - \xi)^4 3[1, 3][[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{13}^{-4} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^3 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3])([1, 3][2, [1, 3]] \\
&\quad \quad - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad + p_{23}^{-7} p_{12}^{-1} p_{13}^{-5} p_{21}^{-5} (1 - \xi)^4 [[[1, 2], [[1, 3], 2]], [1, 3]] 3[1, 3] \\
&:= F,
\end{aligned}$$

$$\begin{aligned}
\langle y_3, F \rangle &= p_{21}^{-2} p_{23}^{-3} p_{13}^{-1} (1 - \xi)^4 [1, 3][[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{31}^{-3} p_{32}^{-2} p_{23}^{-7} p_{12}^{-1} p_{13}^{-5} p_{21}^{-5} (1 - \xi)^4 [[[1, 2], [[1, 3], 2]], [1, 3]][1, 3] \\
&= p_{21}^{-2} p_{23}^{-3} p_{13}^{-1} (1 - \xi)^4 ([1, 3][[1, 2], [[1, 3], 2]], [1, 3]] \\
&\quad + p_{12}^{-1} p_{13}^{-1} p_{21}^{-3} p_{23}^{-2} [[[1, 2], [[1, 3], 2]], [1, 3]][1, 3]) \\
&= 0 \text{ since } [[[1, 2], [[1, 3], 2]], [1, 3]][1, 3] = 0,
\end{aligned}$$

$$\begin{aligned}
\langle y_1, F \rangle &= -p_{21}^{-2} p_{23}^{-3} p_{13}^{-3} (1 - \xi)^4 3[1, 3] p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 \\
&\quad + p_{13}^{-4} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^3 ((p_{13}^{-1} - p_{31}) 3[2, [1, 3]] + p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]] (p_{13}^{-1} - p_{31}) 3) \\
&\quad \quad ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad + p_{12}^{-1} p_{13}^{-2} p_{21}^{-4} p_{23}^{-4} \xi^2 (1 - \xi)^3 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad \quad ((p_{13}^{-1} - p_{31}) 3[2, [1, 3]] + p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]] (p_{13}^{-1} - p_{31}) 3) \\
&\quad + p_{23}^{-7} p_{12}^{-1} p_{13}^{-5} p_{21}^{-5} (1 - \xi)^4 p_{12}^{-2} p_{13}^{-2} \xi (1 - \xi) [2, [1, 3]]^2 3[1, 3] \\
&= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 3[1, 3][2, [1, 3]]^2 \\
&\quad + p_{13}^{-5} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 (3[2, [1, 3]] + p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]] 3) \\
&\quad \quad ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&\quad + p_{12}^{-1} p_{13}^{-7} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3])(3[2, [1, 3]] \\
&\quad \quad + p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]] 3) \\
&\quad + p_{23}^{-7} p_{13}^{-7} p_{21}^{-2} (1 - \xi)^5 \xi [2, [1, 3]]^2 3[1, 3] \\
&:= G,
\end{aligned}$$

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$$\begin{aligned}
< y_3, G > &= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 [1, 3][2, [1, 3]]^2 \\
&+ p_{13}^{-5} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 ([2, [1, 3]] - p_{31}^{-1} p_{32}^{-1} p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]]) \\
&([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&+ p_{12}^{-1} p_{13}^{-7} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 p_{31}^{-2} p_{32}^{-1} ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&([2, [1, 3]] - p_{31}^{-1} p_{32}^{-1} p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} [2, [1, 3]]) \\
&+ p_{31}^{-2} p_{32}^{-2} p_{23}^{-7} p_{13}^{-7} p_{21}^{-2} (1 - \xi)^5 \xi [2, [1, 3]]^2 [1, 3] \\
&= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 [1, 3][2, [1, 3]]^2 \\
&+ p_{13}^{-5} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^5 [2, [1, 3]]([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&+ p_{13}^{-5} p_{23}^{-3} \xi^2 (1 - \xi)^5 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3])[2, [1, 3]] \\
&+ p_{23}^{-5} p_{13}^{-5} p_{21}^{-2} (1 - \xi)^5 \xi^2 [2, [1, 3]]^2 [1, 3] \\
&= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 [1, 3][2, [1, 3]]^2 + p_{13}^{-5} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^5 [2, [1, 3]][1, 3][2, [1, 3]] \\
&- p_{13}^{-5} p_{21}^{-2} p_{23}^{-5} \xi^2 (1 - \xi)^5 [2, [1, 3]]^2 [1, 3] + p_{13}^{-5} p_{23}^{-3} \xi^2 (1 - \xi)^5 [1, 3][2, [1, 3]]^2 \\
&- p_{13}^{-5} p_{23}^{-4} \xi^2 (1 - \xi)^5 p_{21}^{-1} [2, [1, 3]][1, 3][2, [1, 3]] + p_{23}^{-5} p_{13}^{-5} p_{21}^{-2} (1 - \xi)^5 \xi^2 [2, [1, 3]]^2 [1, 3] = 0,
\end{aligned}$$

$$\begin{aligned}
< y_2, G > &= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 p_{21}^{-1} p_{23}^{-2} 3[1, 3] p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21}) \\
&([1, 3][2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&+ p_{13}^{-5} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 (p_{23}^{-1} 3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21})[1, 3]) \\
&+ p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21})[1, 3] 3([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&- p_{12}^{-1} p_{13}^{-7} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^4 p_{21}^{-2} p_{23}^{-2} ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&(p_{23}^{-1} 3 p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21})[1, 3] + p_{21}^{-1} p_{23}^{-1} p_{12}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21})[1, 3] 3) \\
&+ p_{23}^{-7} p_{13}^{-7} p_{21}^{-2} (1 - \xi)^5 \xi p_{21}^{-1} p_{23}^{-1} (1 - p_{12} p_{21})([1, 3][2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) 3[1, 3] \\
&= -p_{23}^{-3} p_{13}^{-5} (1 - \xi)^5 \xi^2 p_{21}^{-1} 3[1, 3]([1, 3][2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&+ p_{13}^{-5} p_{21}^{-2} p_{23}^{-6} \xi^2 (1 - \xi)^6 3[1, 3]([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) \\
&- p_{12}^{-1} p_{13}^{-7} p_{21}^{-1} p_{23}^{-4} \xi^2 (1 - \xi)^6 ([1, 3][2, [1, 3]] - p_{21}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) 3[1, 3] \\
&+ p_{23}^{-8} p_{13}^{-7} p_{21}^{-3} (1 - \xi)^5 \xi (1 - p_{12} p_{21})([1, 3][2, [1, 3]] + p_{21}^{-1} p_{22}^{-1} p_{23}^{-1} [2, [1, 3]][1, 3]) 3[1, 3] = 0.
\end{aligned}$$

Then $[[[1, 2], [1, 3], 2]], [1, 3]]^2 = 0$.

(iii) Assume generalized Dynkin diagram of braided vector space V is the following condition:

$$\begin{array}{c}
q \quad q^{-1} \quad -1 \quad r^{-1} \quad r \\
\bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet
\end{array}, \quad q \in R_m, r \in R_{m'}, q \neq r; m, m' > 1.$$

Then $D(V) = \{[x_1]; [x_2]; [x_3]; [x_1, x_2]; [x_1, x_3]; [[x_1, x_3], x_2]; [[x_1, x_2], [x_1, x_3]]\}$ by Theorem 3.3(i) in [15]. Now we show that every hard super-letter is nilpotent.

$$\begin{aligned}
< y_1, [1, 3][1, 2] > &= (p_{13}^{-1} - p_{31}) 3[1, 2] + p_{11}^{-1} p_{13}^{-1} (p_{12}^{-1} - p_{21}) [1, 3] 2, \\
< y_2 y_1, [1, 3][1, 2] > &= p_{11}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} (p_{12}^{-1} - p_{21}) [1, 3], \\
< y_3 y_1, [1, 3][1, 2] > &= (p_{13}^{-1} - p_{31}) [1, 2], \\
< y_1, [1, 2][1, 3] > &= (p_{12}^{-1} - p_{21}) 2[1, 3] + p_{11}^{-1} p_{12}^{-1} (p_{13}^{-1} - p_{31}) [1, 2] 3, \\
< y_2 y_1, [1, 2][1, 3] > &= (p_{12}^{-1} - p_{21}) [1, 3],
\end{aligned}$$

$$\begin{aligned}
& \langle y_3 y_1, [1, 2][1, 3] \rangle = p_{11}^{-1} p_{12}^{-1} p_{31}^{-1} p_{32}^{-1} (p_{13}^{-1} - p_{31}) [1, 2], \\
& [[1, 2], [1, 3]]^k = ([1, 3][1, 2])^k + (-p_{11} p_{12} p_{31} p_{32})^k ([1, 2][1, 3])^k, \\
& \langle y_1, ([1, 3][1, 2])^k \rangle = \sum_{i=0}^{k-1} (p_{13} p_{12})^{-i} ([1, 3][1, 2])^i \langle y_1, [1, 3][1, 2] \rangle ([1, 3][1, 2])^{k-1-i} \\
& = \sum_{i=0}^{k-1} (p_{13} p_{12})^{-i} ([1, 3][1, 2])^i \{ (p_{13}^{-1} - p_{31}) 3[1, 2] \\
& \quad + p_{11}^{-1} p_{13}^{-1} (p_{12}^{-1} - p_{21}) [1, 3] 2 \} ([1, 3][1, 2])^{k-1-i}, \\
& \langle y_3 y_1, ([1, 3][1, 2])^k \rangle \\
& = \sum_{i=0}^{k-1} (p_{13} p_{12})^{-i} (p_{31}^2 p_{33} p_{32})^{-i} ([1, 3][1, 2])^i ((p_{13}^{-1} - p_{31}) [1, 2]) ([1, 3][1, 2])^{k-1-i} \\
& = (p_{13}^{-1} - p_{31}) [1, 2] ([1, 3][1, 2])^{k-1}, \\
& \langle y_2 y_1, ([1, 3][1, 2])^k \rangle \\
& = \sum_{i=0}^{k-1} (p_{13} p_{12})^{-i} (p_{21}^2 p_{23} p_{22})^{-i} ([1, 3][1, 2])^i (p_{11}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} (p_{12}^{-1} - p_{21}) [1, 3]) ([1, 3][1, 2])^{k-1-i} \\
& = (p_{13} p_{12})^{-n+1} (p_{21}^2 p_{23} p_{22})^{-n+1} ([1, 3][1, 2])^{k-1} (p_{11}^{-1} p_{13}^{-1} p_{21}^{-1} p_{23}^{-1} (p_{12}^{-1} - p_{21}) [1, 3]) \\
& = -(p_{13} p_{21} p_{23})^{-n} (p_{12}^{-1} - p_{21}) ([1, 3][1, 2])^{k-1} [1, 3], \\
& \langle y_1, ([1, 2][1, 3])^k \rangle = \sum_{i=0}^{k-1} (p_{12} p_{13})^{-i} ([1, 2][1, 3])^i \langle y_1, [1, 2][1, 3] \rangle ([1, 2][1, 3])^{k-1-i} \\
& = \sum_{i=0}^{k-1} (p_{12} p_{13})^{-i} ([1, 2][1, 3])^i ((p_{12}^{-1} - p_{21}) 2[1, 3] + p_{11}^{-1} p_{12}^{-1} (p_{13}^{-1} - p_{31}) [1, 2] 3) ([1, 2][1, 3])^{k-1-i}, \\
& \langle y_2 y_1, ([1, 2][1, 3])^k \rangle \\
& = \sum_{i=0}^{k-1} (p_{12} p_{13})^{-i} (p_{21}^2 p_{22} p_{23})^{-i} ([1, 2][1, 3])^i ((p_{12}^{-1} - p_{21}) [1, 3]) ([1, 2][1, 3])^{k-1-i} \\
& = (p_{12}^{-1} - p_{21}) [1, 3] ([1, 2][1, 3])^{k-1}, \\
& \langle y_3 y_1, ([1, 2][1, 3])^k \rangle \\
& = \sum_{i=0}^{k-1} (p_{12} p_{13})^{-i} (p_{31}^2 p_{32} p_{33})^{-i} ([1, 2][1, 3])^i (p_{11}^{-1} p_{12}^{-1} p_{31}^{-1} p_{32}^{-1} (p_{13}^{-1} - p_{31}) [1, 2]) ([1, 2][1, 3])^{k-1-i} \\
& = (p_{12} p_{13})^{-n+1} (p_{31}^2 p_{32} p_{33})^{-n+1} ([1, 2][1, 3])^{k-1} (p_{11}^{-1} p_{12}^{-1} p_{31}^{-1} p_{32}^{-1} (p_{13}^{-1} - p_{31}) [1, 2]) \\
& = -(p_{12} p_{31} p_{32})^{-n} (p_{13}^{-1} - p_{31}) ([1, 2][1, 3])^{k-1} [1, 2],
\end{aligned}$$

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$$\begin{aligned}
& \langle y_1, [1, 2]([1, 3][1, 2])^{k-1} \rangle = (p_{12}^{-1} - p_{21})2([1, 3][1, 2])^{k-1} + p_{11}^{-1}p_{12}^{-1}[1, 2] \\
& \quad \sum_{i=0}^{k-2} (p_{13}p_{12})^{-i}([1, 3][1, 2])^i((p_{13}^{-1} - p_{31})3[1, 2] + p_{11}^{-1}p_{13}^{-1}(p_{12}^{-1} - p_{21})[1, 3]2)([1, 3][1, 2])^{k-2-i}, \\
& \langle y_2y_1, [1, 2]([1, 3][1, 2])^{k-1} \rangle = (p_{12}^{-1} - p_{21})([1, 3][1, 2])^{k-1} + p_{11}^{-1}p_{12}^{-1}[1, 2] \\
& \quad \sum_{i=0}^{k-2} (p_{21}^2p_{22}p_{23})^{-i}(p_{13}p_{12})^{-i}([1, 3][1, 2])^i(p_{11}^{-1}p_{13}^{-1}p_{21}^{-1}p_{23}^{-1}(p_{12}^{-1} - p_{21})[1, 3])([1, 3][1, 2])^{k-2-i} \\
& \quad = (p_{12}^{-1} - p_{21})([1, 3][1, 2])^{k-1} + p_{11}^{-1}p_{12}^{-1}[1, 2](p_{21}^2p_{22}p_{23})^{-n+2}(p_{13}p_{12})^{-n+2} \\
& \quad \quad ([1, 3][1, 2])^{k-2}p_{11}^{-1}p_{13}^{-1}p_{21}^{-1}p_{23}^{-1}(p_{12}^{-1} - p_{21})[1, 3] \\
& \quad = (p_{12}^{-1} - p_{21})([1, 3][1, 2])^{k-1} + (p_{21}p_{13}p_{23})^{-n+1}p_{12}^{-1}(p_{12}^{-1} - p_{21})([1, 2][1, 3])^{k-1}, \\
& \langle y_3y_1, [1, 2]([1, 3][1, 2])^{k-1} \rangle = 0, \\
& \langle y_1, [1, 3]([1, 2][1, 3])^{k-1} \rangle = (p_{13}^{-1} - p_{31})3([1, 2][1, 3])^{k-1} + p_{11}^{-1}p_{13}^{-1}[1, 3] \\
& \quad \sum_{i=0}^{k-2} (p_{12}p_{13})^{-i}([1, 2][1, 3])^i((p_{12}^{-1} - p_{21})2[1, 3] + p_{11}^{-1}p_{12}^{-1}(p_{13}^{-1} - p_{31})[1, 2]3)([1, 2][1, 3])^{k-2-i}, \\
& \langle y_3y_1, [1, 3]([1, 2][1, 3])^{k-1} \rangle = (p_{13}^{-1} - p_{31})([1, 2][1, 3])^{k-1} + p_{11}^{-1}p_{13}^{-1}[1, 3] \\
& \quad \sum_{i=0}^{k-2} (p_{31}^2p_{33}p_{32})^{-i}(p_{12}p_{13})^{-i}([1, 2][1, 3])^i(p_{11}^{-1}p_{12}^{-1}p_{31}^{-1}p_{32}^{-1}(p_{13}^{-1} - p_{31})[1, 2])([1, 2][1, 3])^{k-2-i} \\
& \quad = (p_{13}^{-1} - p_{31})([1, 2][1, 3])^{k-1} \\
& \quad + p_{11}^{-1}p_{13}^{-1}[1, 3](p_{31}^2p_{33}p_{32})^{-n+2}(p_{12}p_{13})^{-n+2}([1, 2][1, 3])^{k-2}p_{11}^{-1}p_{12}^{-1}p_{31}^{-1}p_{32}^{-1}(p_{13}^{-1} - p_{31})[1, 2] \\
& \quad = (p_{13}^{-1} - p_{31})([1, 2][1, 3])^{k-1} + (p_{31}p_{12}p_{32})^{-n+1}p_{13}^{-1}(p_{13}^{-1} - p_{31})([1, 3][1, 2])^{k-1}, \\
& \langle y_2y_1, [1, 3]([1, 2][1, 3])^{k-1} \rangle = 0, \\
& \langle y_2y_1y_2y_1, ([1, 2][1, 3])^k \rangle = 0, \\
& \langle y_2y_1y_2y_1, ([1, 3][1, 2])^k \rangle = 0, \\
& \langle y_3y_1y_3y_1, ([1, 2][1, 3])^k \rangle = 0, \\
& \langle y_3y_1y_3y_1, ([1, 3][1, 2])^k \rangle = 0, \\
& \langle y_2y_1y_3y_1, ([1, 2][1, 3])^k \rangle = \langle y_2y_1, -(p_{12}p_{31}p_{32})^{-n}(p_{13}^{-1} - p_{31})([1, 2][1, 3])^{k-1}[1, 2] \rangle \\
& \quad = -(p_{12}p_{31}p_{32})^{-n}(p_{13}^{-1} - p_{31}) \langle y_2y_1, ([1, 2][1, 3])^{k-1}[1, 2] \rangle, \\
& \quad \langle y_2y_1y_3y_1, ([1, 3][1, 2])^k \rangle = \langle y_2y_1, (p_{13}^{-1} - p_{31})[1, 2]([1, 3][1, 2])^{k-1} \rangle \\
& \quad = (p_{13}^{-1} - p_{31}) \langle y_2y_1, [1, 2]([1, 3][1, 2])^{k-1} \rangle, \\
& \langle y_3y_1y_2y_1, ([1, 3][1, 2])^k \rangle = \langle y_3y_1, -(p_{13}p_{21}p_{23})^{-n}(p_{12}^{-1} - p_{21})([1, 3][1, 2])^{k-1}[1, 3] \rangle \\
& \quad = -(p_{13}p_{21}p_{23})^{-n}(p_{12}^{-1} - p_{21}) \langle y_3y_1, ([1, 3][1, 2])^{k-1}[1, 3] \rangle,
\end{aligned}$$

$$\begin{aligned} < y_3 y_1 y_2 y_1, ([1, 2][1, 3])^k > = < y_3 y_1, (p_{12}^{-1} - p_{21})[1, 3]([1, 2][1, 3])^{k-1} > \\ &= (p_{12}^{-1} - p_{21}) < y_3 y_1, [1, 3]([1, 2][1, 3])^{k-1} >, \end{aligned}$$

$$\begin{aligned} < y_2 y_1 y_3 y_1, [[1, 2], [1, 3]]^k > &= < y_2 y_1 y_3 y_1, ([1, 3][1, 2])^k + (-p_{11} p_{12} p_{31} p_{32})^k ([1, 2][1, 3])^k > \\ &= < y_2 y_1 y_3 y_1, ([1, 3][1, 2])^k > + (p_{12} p_{31} p_{32})^k < y_2 y_1 y_3 y_1, ([1, 2][1, 3])^k > \\ &= (p_{13}^{-1} - p_{31}) < y_2 y_1, [1, 2]([1, 3][1, 2])^{k-1} > \\ &\quad - (p_{12} p_{31} p_{32})^k (p_{12} p_{31} p_{32})^{-n} (p_{13}^{-1} - p_{31}) < y_2 y_1, ([1, 2][1, 3])^{k-1} [1, 2] > = 0, \end{aligned}$$

$$\begin{aligned} < y_3 y_1 y_2 y_1, [[1, 2], [1, 3]]^k > &= < y_3 y_1 y_2 y_1, ([1, 3][1, 2])^k + (-p_{11} p_{12} p_{31} p_{32})^k ([1, 2][1, 3])^k > \\ &= < y_3 y_1 y_2 y_1, ([1, 3][1, 2])^k > + (p_{12} p_{31} p_{32})^k < y_3 y_1 y_2 y_1, ([1, 2][1, 3])^k > \\ &= -(p_{13} p_{21} p_{23})^{-n} (p_{12}^{-1} - p_{21}) < y_3 y_1, ([1, 3][1, 2])^{k-1} [1, 3] > \\ &\quad + (p_{12} p_{31} p_{32})^k (p_{12}^{-1} - p_{21}) < y_3 y_1, [1, 3]([1, 2][1, 3])^{k-1} > \\ &= (p_{13} p_{21} p_{23})^{-n} ((p_{22} p_{33})^{-n} - 1) (p_{12}^{-1} - p_{21}) < y_3 y_1, ([1, 3][1, 2])^{k-1} [1, 3] > . \end{aligned}$$

Then $[[1, 2], [1, 3]]^k = 0$, $k = \frac{m^2 m'^2}{(m, m')}$. \square

Lemma 6.3. (*Theorem 3.1(1) in [3], and [6]*) *There does not exist any m -infinity element in $\mathfrak{B}(V)$ when $\Delta(\mathfrak{B}(V))$ is an arithmetic root system and V is of finite Cartan type with $\text{ord}(q_{ij}) < \infty$ for $1 \leq i, j \leq n$ or $\dim V = n < 3$.*

6.2. The hard super-letters of $\mathbb{B}(V)$ with $\dim V = 2$

We give $D := \{u_1, u_2, \dots, u_r\}$ for all hard super-letters of connected braided vector space V of diagonal type with rank 2 and height h_i for hard super-letter u_i using Definition 2 and Appendix A in [6]. Let $a = 112, b = 122, c = 1112, d = 11212, e = 11112, f = 1112112, g = 1121212, h = 111112, i = 111121112, j = 11121112112, k = 1112112112, l = 11211212, m = 112121121212, n = 112121212, p = 11211211212, q = 1121121211212, r = 12$. $h_u = \text{ord}(p_{uu})$ for any $u \in D(V)$.

- T2. $D = \{1, 2, r\}$.
- T3. $D = \{1, 2, r, a\}$.
- T4. $D = \{1, 2, r, b, a\}$.
- T5. $D = \{1, 2, r, a, d\}$.
- T6. $D = \{1, 2, r, b, a, d\}$.
- T7. $D = \{1, 2, r, a, c\}$.
- T8. $D = \{1, 2, r, a, d, c\}$.
- T9. $D = \{1, 2, r, a, d, g\}$.
- T10. $D = \{1, 2, r, b, a, d, g, c\}$.
- T11. $D = \{1, 2, r, a, d, g, l, c\}$.
- T12. $D = \{1, 2, r, b, a, d, c, f\}$.
- T13. $D = \{1, 2, r, a, d, l, c, f\}$.
- T14. $D = \{1, 2, r, a, c, e\}$.
- T15. $D = \{1, 2, r, a, d, g, c, e\}$.

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T16. $D = \{1, 2, r, a, d, c, f, e\}$.

T17. $D = \{1, 2, r, a, d, g, n, l\}$.

T18. $D = \{1, 2, r, a, d, g, l, q\}$

T19. $D = \{1, 2, r, a, d, g, n, m, l, q, p, c\}$.

T20 $D = \{1, 2, r, a, c, f, j, e\}$.

T21 $D = \{1, 2, r, a, c, f, e, h\}$.

T22 $D = \{1, 2, r, a, d, c, f, k, j, e, i, h\}$.

(Row 2) T2 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 3\} = \{q, q, q\}$.

(Row 3) T2 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 3\} = \{-1, -1, q\}$.

T2 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 3\} = \{-1, q, -1\}$.

(Row 4) T3 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{q^2, q, q^2, q\}$.

(Row 5) T3 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{-1, -q^{-1}, -1, q\}$.

T3 (2) Table A.1 in [6]: we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{-1, q, -1, -q^{-1}\}$.

(Row 6) T3 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{q, \xi, \xi q^{-1}, \xi\}$.

T3 (2) Table A.1 in [6]: we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{\xi q^{-1}, \xi, q, \xi\}$.

(Row 7) T3 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{-1, \xi^{-1}, -1, \xi\}$.

T3 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 4\} = \{-1, \xi, -1, \xi^{-1}\}$.

(Row 8) T4 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [b], u_3 = [r], u_4 = [a], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-\xi^2, -1, -\xi^3, -1, -\xi^{-2}\}$.

T5 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^3, -1, -\xi^2, -\xi^{-2}\}$.

T5 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^3, -1, -\xi^{-2}, -\xi^2\}$.

T7 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [c], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^{-2}, -\xi^2, -1, -\xi^3\}$.

T7 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [c], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^2, -\xi^{-2}, -1, -\xi^3\}$.

(Row 9) T4 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [b], u_3 = [r], u_4 = [a], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-\xi^2, -1, -\xi^{-1}, -1, -\xi^2\}$.

T5 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^{-1}, -1, -\xi^2, -\xi^2\}$.

T7 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [c], u_5 = x_1$.

$\{p_{u_i, u_i} \mid 1 \leq i \leq 5\} = \{-1, -\xi^2, -\xi^2, -1, -\xi^{-1}\}$.

(Row 10) T6 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [b], u_3 = [r], u_4 = [d], u_5 = [a], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{\xi^3, -1, -\xi^2, -1, \xi^3 - \xi\}$.

T9 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [a], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{-1, -\xi^2, -1, \xi^3, -\xi, \xi^3\}$.

T14 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [c], u_5 = [e], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{-1, \xi^3, -\xi, \xi^3, -1, -\xi^2\}$.

(Row 11) T8 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [c], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{q^3, q, q^3, q, q^3, q\}$.

(Row 12) T8 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [c], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{\xi^{-1}, \xi^2, -1, \xi, -1, \xi^2\}$.

T8 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [c], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{-1, \xi, -1, \xi^2, \xi^{-1}, \xi^2\}$.

T8 (3) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [c], u_6 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 6\} = \{-1, \xi^2, \xi^{-1}, \xi^2, -1, \xi\}$.

(Row 13) T10 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [b], u_3 = [r], u_4 = [g], u_5 = [d], u_6 = [a], u_7 = [c], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-\xi^{-4}, -1, \xi, -1, -\xi^{-4}, \xi^6, \xi^{-1}, \xi^6\}$.

T13 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [l], u_5 = [a], u_6 = [f], u_7 = [c], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{\xi^{-1}, \xi^6, -\xi^{-4}, -1, \xi, -1, -\xi^{-4}, \xi^6\}$.

T17 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [n], u_4 = [g], u_5 = [d], u_6 = [l], u_7 = [a], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, \xi, -1, -\xi^{-4}, \xi^6, \xi^{-1}, \xi^6, -\xi^{-4}\}$.

T21 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [f], u_5 = [c], u_6 = [e], u_7 = [h], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, -\xi^{-4}, \xi^6, \xi^{-1}, \xi^6, -\xi^{-4}, -1, \xi\}$.

(Row 14) T11 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [l], u_6 = [a], u_7 = [c], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, -\xi^{-2}, -1, \xi, -1, -\xi^{-2}, -1, \xi\}$.

T16 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [f], u_6 = [c], u_7 = [e], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, \xi, -1, -\xi^{-2}, -1, \xi, -1, -\xi^{-2}\}$.

(Row 15) T11 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [l], u_6 = [a], u_7 = [c], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}, -1, \xi\}$.

T11 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [l], u_6 = [a], u_7 = [c], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, -\xi^{-2}, -1, \xi, -1, -\xi^{-2}, -1, -\xi\}$.

T16 (1) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [f], u_6 = [c], u_7 = [e], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, \xi, -1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}\}$.

T16 (2) Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [f], u_6 = [c], u_7 = [e], u_8 = x_1$. $\{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, -\xi, -1, -\xi^{-2}, -1, \xi, -1, -\xi^{-2}\}$.

(Row 16) T12 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [b], u_3 = [r], u_4 =$

$[d], u_5 = [a], u_6 = [f], u_7 = [c], u_8 = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{\xi^5, -1, \xi^3, -\xi^{-4}, \xi^3, -1, \xi^5, -\xi\}.$

T15 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [a], u_6 = [c], u_7 = [e], u_8 = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-\xi^{-4}, \xi^3, -1, \xi^5, -\xi, \xi^5, -\xi, \xi^5, -1, \xi^3\}.$

T18 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [g], u_4 = [d], u_5 = [q], u_6 = [l], u_7 = [a], u_8 = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, \xi^3, -\xi^{-4}, \xi^3, -1, \xi^5, -\xi, \xi^5\}.$

T20 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [a], u_4 = [f], u_5 = [j], u_6 = [c], u_7 = [e], u_8 = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 8\} = \{-1, \xi^5, -\xi, \xi^5, -1, \xi^3, -\xi^{-4}, \xi^3\}.$

(Row 17) T19 Table A.1 in [6] : we set $u_1 = x_2, u_2 = [r], u_3 = [n], u_4 = [g], u_5 = [m], u_6 = [d], u_7 = [q], u_8 = [l], u_9 = [p], u_{10} = [a], u_{11} = [c], u_{12} = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 12\} = \{-1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}, -1, -\xi\}.$

T22 Table A.1 in [6]: we set $u_1 = x_2, u_2 = [r], u_3 = [d], u_4 = [a], u_5 = [k], u_6 = [f], u_7 = [j], u_8 = [c], u_9 = [i], u_{10} = [e], u_{11} = [h], u_{12} = x_1. \{p_{u_i, u_i} \mid 1 \leq i \leq 12\} = \{-1, -\xi, -1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}, -1, -\xi, -1, -\xi^{-2}\}.$

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